Chapter 2 Multivariate Distributions

Section 2.5 Independent Random Variables

Joint Density: $f(X_1, X_2)$. Marginal pdfs: $f_1(X_1)$ and $f_2(X_2)$.

$$f(X_1, X_2) = \begin{cases} f(X_2|X_1)\, f_1(X_1) \\ or \\ f(X_1|X_2)\, f_2(X_2) \end{cases}$$

Now suppose that $f(X_2|X_1)$ does not depend on X_1 .

Consider $f_2(X_2) = \int_{-\infty}^{\infty} f(X_1, X_2)\, dx_1 = \int_{-\infty}^{\infty} f(X_2|X_1)\, f_1(X_1)\, dx_1$ so,

$$f_2(X_2) = f(X_2|X_1) \int_{-\infty}^{\infty} f_1(X_1)\, dx_1 = f(X_2|X_1) .$$

Hence, if $f(X_1, X_2) = f(X_2|X_1)\, f_1(X_1)$ and $f_2(X_2) = f(X_2|X_1)$ then $f(X_1, X_2) = f_1(X_1)\, f_2(X_2)$.

Note: The same discussion applies to the discrete random variables.

<u>Definition (Independence).</u> Let the random variables X_1 and X_2 have the joint pdf $f(X_1, X_2)$ and the marginal pdfs $f_1(X_1)$ and $f_2(X_2)$, respectively. The random variables X_1 and X_2 are said to be independent if, and only if, $f(X_1, X_2) = f_1(X_1)\, f_2(X_2)$. Otherwise they are said to be dependent.

Note: There may be certain points $(x_1, x_2) \in \mathcal{A}$ at which $f(X_1, X_2) \neq f_1(X_1)\, f_2(X_2)$. However, if $\boldsymbol{A}$ is the set of points (x_1, x_2) at which the equality does not hold, then $P(A) = 0$.

<u>Example 1:</u> $f(X_1, X_2) = e^{-x_1 - x_2};\ 0 < x_1 < \infty,\ 0 < x_2 < \infty$.

$f(X_1, X_2) = e^{-x_1} . e^{-x_2} = f_1(X_1)\, f_2(X_2)$, where $\begin{matrix} f_1(X_1) = e^{-x_1};\ 0 < x_1 < \infty \\ f_2(X_2) = e^{-x_2};\ 0 < x_2 < \infty \end{matrix}$.

Example 2: $f(X_1, X_2) = x_1 + x_2;\ \ 0 < x_1 < 1\ ;\ \ 0 < x_2 < 1$.

$f(X_1, X_2) = x_1 + x_2 \neq \left(x_1 + \frac{1}{2}\right)\left(x_2 + \frac{1}{2}\right) = f_1(X_1)\, f_2(X_2)$, where $\begin{array}{l} f_1(X_1) = x_1 + \frac{1}{2};\ \ 0 < x_1 < 1 \\ f_2(X_2) = x_2 + \frac{1}{2};\ \ 0 < x_2 < 1 \end{array}$.

Theorem 2.5.1: Let the random variables X_1 and X_2 have the joint pdf $f(X_1, X_2)$. Then X_1 and X_2 are independent if and only if $f(X_1, X_2)$ can be written as a product of a non-negative function of x_1 alone and a product of a non-negative function of x_2 alone. That is, $f(X_1, X_2) = g(X_1)\, h(X_2)$, where $g(X_1) > 0;\ x_1 \in \mathcal{A}_1$, and zero elsewhere, and $h(X_2) > 0;\ x_2 \in \mathcal{A}_2$, and zero elsewhere .

Proof:

$\Rightarrow$ If X_1 and X_2 are independent, then

$$f(X_1, X_2) = f_1(X_1)\, f_2(X_2)\ .\ \text{Thus,}\ f(X_1, X_2) = g(X_1) h(X_2).$$

$\Leftarrow$ If $f(X_1, X_2) = g(X_1) h(X_2)$ then

$$f_1(X_1) = \int_{-\infty}^{\infty} f(X_1, X_2)\, dx_2 = \int_{-\infty}^{\infty} g(x_1) h(x_2)\, dx_2 = c_1 g(x_1)$$

$$f_2(X_2) = \int_{-\infty}^{\infty} f(X_1, X_2)\, dx_1 = \int_{-\infty}^{\infty} g(x_1) h(x_2)\, dx_1 = c_2 h(x_2).$$

We already know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X_1, X_2)\, dx_1 dx_2 = 1$. Then,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X_1, X_2)\, dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1) h(x_2)\, dx_1 dx_2 = \left[\int_{-\infty}^{\infty} g(x_1)\, dx_1\right]\left[\int_{-\infty}^{\infty} h(x_2)\, dx_2\right] = c_1 c_2\ .$$

So, $f(X_1, X_2) = g(X_1) h(X_2) = c_1 g(X_1).c_2 h(X_2) = f_1(X_1)\, f_2(X_2)$.

Hence, X_1 and X_2 are independent.

In Example 1: $f(X_1, X_2) = e^{-x_1 - x_2};\ \ 0 < x_1 < \infty,\ 0 < x_2 < \infty$.

$f(X_1, X_2) = e^{-x_1}.\, e^{-x_2} = f_1(X_1)\, f_2(X_2)$, where $\begin{array}{l} f_1(X_1) = e^{-x_1};\ \ 0 < x_1 < \infty \\ f_2(X_2) = e^{-x_2};\ \ 0 < x_2 < \infty \end{array}$.

X_1 and X_2 are independent.

In Example 2: $f(X_1, X_2) = x_1 + x_2;\ \ 0 < x_1 < 1\ ;\ \ 0 < x_2 < 1$.

$f(X_1, X_2) = x_1 + x_2 \neq (x_1 + \frac{1}{2})(x_2 + \frac{1}{2}) = f_1(X_1)\, f_2(X_2)$, where $\begin{matrix} f_1(X_1) = x_1 + \frac{1}{2};\ \ 0 < x_1 < 1 \\ f_2(X_2) = x_2 + \frac{1}{2};\ \ 0 < x_2 < 1 \end{matrix}$.

X_1 and X_2 are dependent.

Example 3: $f(X_1, X_2) = e^{-x_1 - x_2};\ \ 0 < x_1 < x_2 < \infty$.

$f(X_1, X_2) = e^{-x_1} .\, e^{-x_2} = f_1(X_1)\, f_2(X_2)$, where $\begin{matrix} f_1(X_1) = e^{-x_1};\ \ 0 < x_1 < \infty \\ f_2(X_2) = e^{-x_2};\ \ 0 < x_2 < \infty \end{matrix}$.

X_1 and X_2 are dependent since the sample space is not a product space.

If the sample space is not a product space, that is, it's bounded by a curve that is neither horizontal nor a vertical line, then the random variables X_1 and X_2 are dependent.

Theorem 2.5.3: If X_1 and X_2 are independent with marginal pdfs $f_1(X_1)$ and $f_2(X_2)$, respectively, then $P(a < X_1 < b,\ c < X_2 < d) = P(a < X_1 < b).P(c < X_2 < d)$ for every $a < b$ and $c < d$, where $a, b, c,$ and d are constants.

Proof:
Since X_1 and X_2 are independent, then $f(X_1, X_2) = f_1(X_1)\, f_2(X_2)$.

$$P(a < X_1 < b,\ c < X_2 < d) = \int_c^d \int_a^b f(X_1, X_2)\, dx_1 dx_2 = \int_c^d \int_a^b f_1(x_1)\, f_2(x_2)\, dx_1 dx_2$$
$$= \int_a^b f_1(x_1)\, dx_1 \int_c^d f_2(x_2)\, dx_2 = P(a < X_1 < b).P(c < X_2 < d).$$

Theorem 2.5.4: Suppose X_1 and X_2 are independent random variables with marginal pdfs $f_1(X_1)$ and $f_2(X_2)$, respectively. Suppose also, that $E(u(X_1))$ and $E(v(X_2))$ exist. Then,

$$E(u(X_1)\ v(X_2)) = E(u(X_1)).E(v(X_2)).$$

Proof:

$$E(u(X_1)v(X_2)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(X_1)v(X_2)\, f(X_1, X_2)\, dx_1 dx_2$$
$$= \left[\int_{-\infty}^{\infty} u(X_1)\, f_1(X_1) dx_1\right]\left[\int_{-\infty}^{\infty} v(X_2)\, f_2(X_2)\, dx_2\right] = E(u(X_1)).E(v(X_2)) .$$

Example 4: $Cov(X,Y) = E(XY) - \mu_1\mu_2$.

If X and Y are independent then $E(XY) = E(X)E(Y) = \mu_1\mu_2$. Hence, $Cov(X,Y) = E(XY) - \mu_1\mu_2 = E(X)E(Y) - \mu_1\mu_2 = \mu_1\mu_2 - \mu_1\mu_2 = 0$. Since, the covariance is zero the $\rho = \frac{\text{cov}(X,Y)}{\sqrt{\sigma_1^2\sigma_2^2}} = \frac{0}{\sqrt{\sigma_1^2\sigma_2^2}} = 0$.

Theorem 2.5.5: Suppose the joint mgf, $M(t_1,t_2)$, exists for the random variables X_1 and X_2. Then X_1 and X_2 are independent if and only if $M(t_1,t_2) = M(t_1,0)M(0,t_2)$, that is, the joint mgf factors into the product of the marginal mgfs.

Proof:

$\Rightarrow$ If X_1 and X_2 are independent, then

$$M(t_1,t_2) = E\left(e^{t_1x_1+t_2x_2}\right) = E\left(e^{t_1x_1}\right)E\left(e^{t_2x_2}\right) = M(t_1,0)M(0,t_2) \ .$$

$\Leftarrow$ Assume the joint mgf of X_1 and X_2 is given by $M(t_1,t_2) = M(t_1,0)M(0,t_2)$.

Note: $\quad M(t_1,0) = \int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1)\, dx_1 \quad \textit{and} \quad M(0,t_2) = \int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2)\, dx_2$

Thus we have

$$M(t_1,0)M(0,t_2) = \left[\int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1)\, dx_1\right]\left[\int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2)\, dx_2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f_1(x_1)f_2(x_2)\, dx_1dx_2 \,.$$

By assumption $M(t_1,t_2) = M(t_1,0)M(0,t_2)$; so $M(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f_1(x_1)f_2(x_2)\, dx_1dx_2$.

But $M(t_1,t_2)$ is the mgf of X_1 and X_2. Thus also

$$M(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f(X_1,X_2)\, dx_1dx_2 \,.$$

Hence, if $M(t_1,t_2) = M(t_1,0)M(0,t_2)$, then X_1 and X_2 are independent.

HW: 2.5.1 to 2.5.6 pp. 116