Chapter 3 Some Special Distributions

Section 3.2 The Poisson Distribution

Consider the series $1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+\frac{m^4}{4!}+\ldots=\sum_{x=0}^{\infty}\frac{m^x}{x!}$. The series converges to e^m, for all values of m. Now consider the function $f\left(x\right)=\frac{m^x e^{-m}}{x!}$; $x=0,1,2,\ldots$, and zero elsewhere, where $m>0$.

Question: Is $f\left(x\right)$ a probability density function? (Yes)

1. Since $m>0$, then $f(x)\geq 0$ and

2. $\sum_{x=0}^{\infty} f(x)=1$. Since $\sum_{x=0}^{\infty}\frac{m^x e^{-m}}{x!}=e^m e^{-m}=e^0=1$.

$f\left(x\right)$ is called a Poisson Distribution and the random variable X is discrete.

There are many applications that follow a Poisson distribution. For example:

1. Let the random variable X be the number of cars in an intersection in a certain interval of time. Given the appropriate m, X will follow a Poisson distribution.

2. At the grocery store. Let the random variable X be the number of customers served by the express lane at a given interval of time. Given the appropriate m, X will follow a Poisson distribution.

Derivation of the Poisson Distribution

To make the derivation easier, first make an outline of the major steps, and then fill in the details, for each step later.

<u>Step 1:</u> Definition of the terms and listing of assumptions

Let $g\left(x,w\right)=\text{p}\left(x \text{ changes in an interval of lenght } w\right)$ and define $g\left(0,0\right)=1$, $g\left(x,0\right)=0$. Furthermore, let the symbol $\text{O}\left(h\right)$ represent any function such that $\lim_{h\to 0}\frac{\text{O}(h)}{h}=0$, (Thus h^2=$\text{O}\left(h\right)$, h^3=$\text{O}\left(h\right)$, h^2+h^3=$\text{O}\left(h\right)$, $\text{O}\left(h\right)+\text{O}\left(h\right)=\text{O}\left(h\right)$.

The Poisson postulates (or assumptions) are:

a. $g(1,h) = \lambda h + O(h)$, $\lambda > 0$, $h > 0$ where λ is a positive constant. (The probability of one change in a short interval of time h is approximately proportional to the length of the interval)

b. $\sum_{x=2}^{\infty} g(x,h) = O(h)$ (The probability of **two or more** changes in the same short interval of time h is approximately equal to zero)

c. The number of changes in any interval is independent of the number of changes in any non-overlapping interval.

<u>**Step 2:**</u> Show that $g(0,w) = e^{-\lambda w}$ by getting $\frac{dg(0,w)}{dw}$ from $g(0,w+h)$, and $g(0,w)$, and the definition of the derivative.

Note: Before we show **step 2** we need to know the probability of zero changes in a short interval h, i.e. $g(0,h)$. Also note that from **step 1**, parts (a) and (b), we know the following:

$g(1,h) = \lambda h + O(h)$ and $\sum_{x=2}^{\infty} g(x,h) = O(h)$. Hence, $g(1,h) + \sum_{x=2}^{\infty} g(x,h) = \sum_{x=1}^{\infty} g(x,h)$.

Where $\sum_{x=1}^{\infty} g(x,h) = \lambda h + O(h) + O(h) = \lambda h + O(h)$.

Now note that

$$g(0,h) + \sum_{x=1}^{\infty} g(x,h) = 1 \Rightarrow g(0,h) = 1 - \sum_{x=1}^{\infty} g(x,h) \Rightarrow g(0,h) = 1 - (\lambda h + O(h)) \Rightarrow$$

$$g(0,h) = 1 - \lambda h - O(h)$$

Now, we can show **step 2**.

Step 1 part (c)

$$g(0,w+h) = g(0,w)g(0,h) = g(0,w)(1 - \lambda h - O(h)) = g(0,w) - \lambda h g(0,w) - O(h)g(0,w)$$

$g(0,w+h) - g(0,w) = -\lambda h g(0,w) - O(h)g(0,w)$ Now divide both sides by h.

$\frac{g(0,w+h) - g(0,w)}{h} = -\lambda g(0,w) - \frac{O(h)}{h} g(0,w)$ Now take the limit as $h \to 0$.

0

$$\lim_{h\to 0} \frac{g(0,w+h) - g(0,w)}{h} = \lim_{h\to 0}(-\lambda g(0,w)) - \lim_{h\to 0}\left(\frac{O(h)}{h} g(0,w)\right)$$

$$\frac{dg(0,w)}{dw}=-\lambda g(0,w) \Rightarrow \frac{dg(0,w)}{dw}\cdot\frac{1}{g(0,w)}=-\lambda \Rightarrow \frac{d\ln(g(0,w))}{dw}=-\lambda$$ Now

integrate both sides with respect to w .

Step 1: g(0,0)=1

$$\ln(g(0,w))=-\lambda w+C \Rightarrow g(0,w)=e^{-\lambda w+C}$$ For $w=0$ we have

$$g(0,0)=e^{-\lambda\cdot 0+C} \Rightarrow 1=e^{C} \Rightarrow C=0.$$ Hence, $g(0,w)=e^{-\lambda w}$.

Step 3: Show that $\frac{dg(x,w)}{dw}=-\lambda g(x,w)+\lambda g(x-1,w)$ from $g(x,w+h)$, and $g(x,w)$, and the definition of the derivative.

First note that

$$g(x,w+h)=g(x,w)g(0,h)+g(x-1,w)g(1,h)+\overbrace{\{g(x-2,w)g(2,h)+\ldots+g(0,w)g(x,h)\}}^{\text{From Step 1 (b), all the terms sum to the O(h)}}.$$

$$g(x,w+h)=g(x,w)g(0,h)+g(x-1,w)g(1,h)+\mathrm{O}(h)$$

$$g(x,w+h)=g(x,w)[1-\lambda h-\mathrm{O}(h)]+g(x-1,w)[\lambda h+\mathrm{O}(h)]+\mathrm{O}(h)$$

$$g(x,w+h)=g(x,w)-\lambda hg(x,w)-\mathrm{O}(h)g(x,w)+\lambda hg(x-1,w)+\mathrm{O}(h)g(x-1,w)+\mathrm{O}(h)$$

$$g(x,w+h)-g(x,w)=-\lambda hg(x,w)-\mathrm{O}(h)g(x,w)+\lambda hg(x-1,w)+\mathrm{O}(h)g(x-1,w)+\mathrm{O}(h)$$

Now divide both sides by h .

$$\frac{g(x,w+h)-g(x,w)}{h}=-\lambda g(x,w)-\frac{\mathrm{O}(h)}{h}g(x,w)+\lambda g(x-1,w)+\frac{\mathrm{O}(h)}{h}g(x-1,w)+\frac{\mathrm{O}(h)}{h}$$

Now take the limit as $h\to 0$.

$$\lim_{h\to 0}\frac{g(x,w+h)-g(x,w)}{h}=-\lim_{h\to 0}\lambda g(x,w)-\lim_{h\to 0}\underbrace{\frac{\mathrm{O}(h)}{h}}_{0}g(x,w)+\lim_{h\to 0}\lambda g(x-1,w)+\lim_{h\to 0}\underbrace{\frac{\mathrm{O}(h)}{h}}_{0}g(x-1,w)+\lim_{h\to 0}\underbrace{\frac{\mathrm{O}(h)}{h}}_{0}$$

$$\frac{dg(x,w)}{dw}=-\lambda g(x,w)+\lambda g(x-1,w)$$

Step 4: Express $e^{\lambda w}g(x,w)$ as an integral using the result in **Step 3** .

$$\frac{dg(x,w)}{dw}=-\lambda g(x,w)+\lambda g(x-1,w) \Rightarrow \frac{dg(x,w)}{dw}+\lambda g(x,w)=\lambda g(x-1,w)$$

Now multiply both sides by $e^{\lambda w}$.

$$\Rightarrow e^{\lambda w}\frac{dg(x,w)}{dw}+\lambda e^{\lambda w}g(x,w)=\lambda e^{\lambda w}g(x-1,w)$$

$$\Rightarrow \frac{d\left(e^{\lambda w}g(x,w)\right)}{dw}=\lambda e^{\lambda w}g(x-1,w)$$ Now integrate both sides from 0 to w.

$$\Rightarrow e^{\lambda w}g(x,w)=\int_0^w \lambda e^{\lambda w}g(x-1,w)\,dw.$$

Step 5: Show that $g(x,w)=\frac{(\lambda w)^x e^{-\lambda w}}{x!}$. Use the result from **Step 4** and mathematical induction to get the final result.

$e^{\lambda w-\lambda w}=e^0=1$

Let $x=1$ $\Rightarrow e^{\lambda w}g(1,w)=\int_0^w \lambda e^{\lambda w}g(1-1,w)\,dw=\int_0^w \lambda e^{\lambda w}g(0,w)\,dw=\int_0^w \lambda \cancel{e^{\lambda w}}\,\cancel{e^{-\lambda w}}\,dw$

$$\Rightarrow e^{\lambda w}g(1,w)=\lambda w \Rightarrow g(1,w)=\lambda w\,e^{-\lambda w} \Rightarrow g(1,w)=\frac{(\lambda w)\,e^{-\lambda w}}{1}.$$

Let $x=2$ $\Rightarrow e^{\lambda w}g(2,w)=\int_0^w \lambda e^{\lambda w}g(2-1,w)\,dw=\int_0^w \lambda e^{\lambda w}g(1,w)\,dw=\int_0^w \lambda \cancel{e^{\lambda w}}\frac{(\lambda w)\,\cancel{e^{-\lambda w}}}{1}\,dw$

$$\Rightarrow e^{\lambda w}g(2,w)=\frac{\lambda^2 w^2}{2.1} \Rightarrow g(2,w)=\frac{\lambda^2 w^2 e^{-\lambda w}}{2.1} \Rightarrow g(2,w)=\frac{(\lambda w)^2\,e^{-\lambda w}}{2.1}.$$

Let $x=3$ $\Rightarrow e^{\lambda w}g(3,w)=\int_0^w \lambda e^{\lambda w}g(3-1,w)\,dw=\int_0^w \lambda e^{\lambda w}g(2,w)\,dw=\int_0^w \lambda \cancel{e^{\lambda w}}\frac{(\lambda w)^2\,\cancel{e^{-\lambda w}}}{2.1}\,dw$

$$\Rightarrow e^{\lambda w}g(3,w)=\frac{\lambda^3 w^3}{3.2.1} \Rightarrow g(3,w)=\frac{\lambda^3 w^3 e^{-\lambda w}}{3.2.1} \Rightarrow g(3,w)=\frac{(\lambda w)^3\,e^{-\lambda w}}{3!}.$$

The formula is correct for $x=0,1,2,3,\ldots$. Now assume that it holds for $x-1$, so that $g(x-1,w)=\frac{(\lambda w)^{x-1}\,e^{-\lambda w}}{(x-1)!}$.

Then $e^{\lambda w}g(x,w)=\int_0^w \lambda e^{\lambda w}g(x-1,w)\,dw=\int_0^w \lambda \cancel{e^{\lambda w}}\frac{(\lambda w)^{x-1}\,\cancel{e^{-\lambda w}}}{(x-1)!}\,dw$

$$\Rightarrow e^{\lambda w}g(x,w)=\frac{\lambda^x w^x}{x.\ldots.3.2.1} \Rightarrow g(x,w)=\frac{\lambda^x w^x e^{-\lambda w}}{x.\ldots.3.2.1} \Rightarrow g(x,w)=\frac{(\lambda w)^x\,e^{-\lambda w}}{x!}.$$

Since the formula holds for $x=2$, it holds for $x=3$. Since it holds for $x=3$, it holds for $x=4$, e.t.c., for all integer x.

The Moment-generating function.

$$M(t)=\sum_{x=0}^{\infty}e^{tx}f(x)=\sum_{x=0}^{\infty}e^{tx}\frac{m^{x}e^{-m}}{x!} \quad \text{where} \quad m=\lambda w$$

$$M(t)=e^{-m}\sum_{x=0}^{\infty}\frac{\left(me^{t}\right)^{x}}{x!}=e^{-m}e^{me^{t}} \quad \Rightarrow M(t)=e^{m(e^{t}-1)} \text{ for real values of } t.$$

$$M'(t)=e^{m(e^{t}-1)}\left(me^{t}\right) \quad \text{and} \quad M''(t)=e^{m(e^{t}-1)}\left(me^{t}\right)+e^{m(e^{t}-1)}\left(me^{t}\right)^{2}.$$

Work for the derivative, $M'(t)$. Let $y=e^{me^{t}-m}$ and $u=me^{t}-m$ then $y=e^{u}$.

$$\frac{dy}{dt}=\frac{dy}{du}.\frac{du}{dt}=\left(e^{u}\right).\left(me^{t}\right)=\left(e^{me^{t}-m}\right)\left(me^{t}\right)=e^{m(e^{t}-1)}\left(me^{t}\right).$$

Now: $\mu=M'(0)=m$ and $\sigma^{2}=M''(0)-\left(M'(0)\right)^{2}=m+m^{2}-m^{2}=m$.

Note: $\mu=\sigma^{2}=m>0$. Hence, $f(x)=\frac{\mu^{x}e^{-\mu}}{x!}$; $x=0,1,2,3,\ldots$, and zero elsewhere.

Also, the Poisson random variable X is denoted by $X \sim \text{Poisson}(\mu)$.

Note: $M(t)=e^{\mu(e^{t}-1)}$

Examples 1, 2, and 3 on pages 152-153.

HW: Learn the derivation of the Poisson distribution.