Chapter 3 Some Special Distributions

Section 3.4 The Normal Distribution

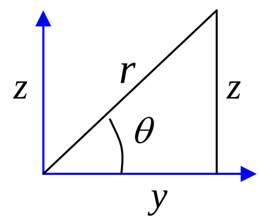
Consider the integral $I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy$. This integral exists since $e^{-\frac{y^2}{2}}$ is continuous and bounded by $0 < e^{-\frac{y^2}{2}} < e^{-|x|+1}$, for $-\infty < y < \infty$ and $\int_{-\infty}^{\infty} e^{-|y|+1}\, dy = 2e$.

To evaluate the integral I consider the following work. Note, that $I > 0$ and

$$I \times I = I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy \right)\left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz \right) \Rightarrow I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{y^2+z^2}{2}}\, dydz$$

. We evaluate this integral by changing to polar coordinates.

$$\sin\theta = \tfrac{z}{r} \ \ and \ \ \cos\theta = \tfrac{y}{r}$$



Let $y = r\cos\theta$ and $z = r\sin\theta$. The Jacobean of the transformation is given by

$$J = \begin{vmatrix} \frac{dy}{dr} & \frac{dy}{d\theta} \\ \frac{dz}{dr} & \frac{dz}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

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$$J = r\cos^2\theta - \left(-r\sin^2\theta\right) = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

Let $u = \frac{r^2}{2}$

Now, $I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{(r\cos\theta)^2 + (r\sin\theta)^2}{2}} |J|\, drd\theta = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}}\, r\, drd\theta = \int_0^{2\pi} \int_0^{\infty} e^{-u}\, du\, d\theta$

$$I^2 = \int_0^{2\pi} 1\ d\theta = 2\pi \quad \Rightarrow I = \pm\sqrt{2\pi}$$

Since $I > 0$ we have $I = \sqrt{2\pi}$.

$$I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy = \sqrt{2\pi} \quad \Rightarrow \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = 1 .$$

Now, introduce a new variable, say $Y = \frac{X-a}{b}$; $\Rightarrow$ $dy = \frac{1}{b}\, dx$ where $b > 0$.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = 1 \;\Rightarrow\; \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(\frac{x-a}{b}\right)^2}{2}}\, dx = 1$$

$f(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(\frac{x-a}{b}\right)^2}{2}}$; where $-\infty < x < \infty$ Note: $f(x)$ satisfies the conditions of being a pdf of the continuous type random variable. It's called the Normal distribution function and it's denoted by $X \sim N(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma)$.

Moment generating function

$$M(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(\frac{x-a}{b}\right)^2}{2}}\, dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2 + tx}\, dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{tx - \frac{x^2 - 2ax + a^2}{2b^2}}\, dx$$

$$M(t) = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{-2b^2tx + x^2 - 2ax + a^2}{2b^2}}\, dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{x^2 - \left(2b^2t + 2a\right)x + a^2}{2b^2}}\, dx$$

Now complete the square $\left(\frac{2b^2t + 2a}{2}\right)^2 = \left(b^2t + a\right)^2$

$$M(t) = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{x^2 - \left(2b^2t + 2a\right)x + \left(b^2t + a\right)^2 - \left(b^2t + a\right)^2 + a^2}{2b^2}}\, dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(x - \left(b^2t + a\right)\right)^2 - \left(b^2t + a\right)^2 + a^2}{2b^2}}\, dx$$

$$M(t) = e^{\frac{\left(b^2t + a\right)^2 - a^2}{2b^2}} \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(x - \left(b^2t + a\right)\right)^2}{2b^2}}\, dx = e^{\frac{\left(b^2t + a\right)^2 - a^2}{2b^2}} = e^{\frac{\left(b^2t + a - a\right)\left(b^2t + a + a\right)}{2b^2}}$$

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$$M(t) = e^{\frac{\left(b^2t\right)\left(b^2t + a + a\right)}{2b^2}} = e^{\frac{b^4t^2 + 2ab^2t}{2b^2}} = e^{\frac{b^2t^2}{2} + at}$$

Hence, $M(t) = e^{\frac{b^2t^2}{2} + at}$.

Now, we would like to find a and b. Work for the derivative, $M'(t)$. Let

$y = e^{\frac{b^2t^2}{2}+at}$ and $u = \frac{b^2t^2}{2} + at$ then $y=e^u$.

$$\frac{dy}{dt} = \frac{dy}{du}.\frac{du}{dt} = (e^u).(a+b^2t) = \left(e^{\frac{b^2t^2}{2}+at}\right)(a+b^2t) = M(t)(a+b^2t).$$

$$M'(t) = M(t)(a+b^2t)\Big|_{t=0} = a \quad . \quad \text{Since, } \mu = M'(0) = a \quad \Rightarrow a = \mu .$$

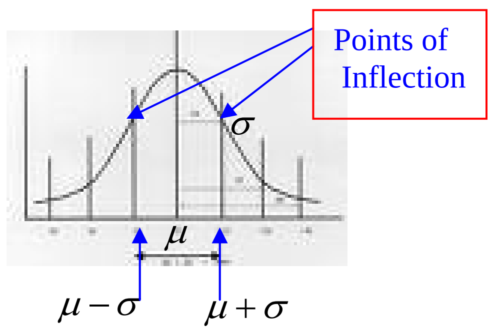
$$M''(t) = M(t)(b^2) + M(t)(a+b^2t)^2\Big|_{t=0} = a^2 + b^2 .$$

Since, $\sigma^2 = M''(0) - (M'(0))^2 = a^2 + b^2 - a^2 = b^2 \Rightarrow b^2 = \sigma^2$.

Now, the probability density function, pdf, and the mgf can be written in terms of μ and σ .

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \; ; \text{ where } -\infty < x < \infty \text{ and } M(t) = e^{\frac{\sigma^2t^2}{2}+\mu t} .$$

If $X \sim N(\mu = 0, \sigma^2 = 1)$ then $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$; where $-\infty < x < \infty$ and it's called the standard normal distribution.



Properties of the Normal distribution:

1. Symmetric about μ .
2. Has the maximum at $x = \mu$ and is $\frac{1}{\sigma\sqrt{2\pi}}$.
3. The points at $x = \mu \pm \sigma$ are points of inflection.

Theorem 0: If the random variable $X \sim N(\mu , \sigma^2)$, then the random variable $W = \frac{X-\mu}{\sigma}$ has a normal distribution with $\mu = 0$ and $\sigma = 1$; i.e $W \sim N(0,1)$.

Proof: $G(W) = P(W \le w) = P\left(\frac{X-\mu}{\sigma} \le w\right) = P(X \le \sigma w + \mu) = \int_{-\infty}^{\sigma w + \mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$.

$$g(w) = G'(W) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\sigma w + \mu - \mu}{\sigma}\right)^2} \cdot \frac{d}{dw}(\sigma w + \mu) - 0 = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(w)^2} \cdot \sigma = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$$

Hence, $g(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$; $-\infty < w < \infty$ and $W \sim N(0,1)$.

If the random variable $X \sim N(\mu , \sigma^2)$, then $P(X = c_1) = 0$ since X is a continuous random variable.

$$P(c_1 \le X \le c_2) = P(X \le c_2) - P(X \le c_1) = P\left(W \le \frac{c_2-\mu}{\sigma}\right) - P\left(W \le \frac{c_1-\mu}{\sigma}\right)$$

Example: If $X \sim N(2, 25)$, compute $P(0 \le X \le 10)$.

$P(0 \le X \le 10) = P(X \le 10) - P(X \le 0) = P\left(W \le \frac{10-2}{5}\right) - P\left(W \le \frac{0-2}{5}\right)$

$P(0 \le X \le 10) = P(W \le 1.6) - P(W \le -0.4) = 0.945 - 0.345 = 0.60$

Example: Compute $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$.

$P(\mu - 2\sigma \le X \le \mu + 2\sigma) = P\left(\frac{\mu-2\sigma-\mu}{\sigma} \le W \le \frac{\mu+2\sigma-\mu}{\sigma}\right) = P(-2 \le W \le 2)$

$= P(W \le 2) - P(W \le -2) = 0.977 - .023 = 0.954$

Example: Suppose that 10 percent of the probability distribution that is $N(\mu , \sigma^2)$ is below 60 and that 5 percent is above 90. Find μ and σ.

We are going to need two equations since we have two unknowns.

1. $P(X \le 60) = 0.10 \Rightarrow P\left(W \le \frac{60-\mu}{\sigma}\right) = 0.10 \Rightarrow \frac{60-\mu}{\sigma} = -1.282$ from the table.
2. $P(X \ge 90) = 0.05 \Rightarrow P\left(W \le \frac{90-\mu}{\sigma}\right) = 0.95 \Rightarrow \frac{60-\mu}{\sigma} = 1.645$ from the table.

Now, solving for μ and σ, we found that $\mu = 73.1$ and $\sigma = 10.2$.

Theorem 1: If the random variable $X \sim N\left(\mu\, ,\sigma^2\right)$, then the random variable $V = \left(\frac{X-\mu}{\sigma}\right)^2$ has a Chi-square distribution with $r = 1$ degrees of freedom ; i.e $V \sim \chi^2_{(r=1)}$.

Proof: Note that $V = W^2$ implying that $V > 0$ and where $W = \frac{X-\mu}{\sigma}$; i.e $W \sim N(0\, ,1)$.

$G(V) = P(V \le v) = P\left(W^2 \le v\right) = P\left(-\sqrt{v} \le W \le \sqrt{v}\right) = 2P\left(0 \le W \le \sqrt{v}\right)$ since the function is even. So, $G(V) = 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\, dw \; ; \; V > 0$

$$g(v) = G^{'}(V) = 2\,\frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{v})^2}{2}}\, \frac{d}{dv}\left(\sqrt{v}\right) - 0 = \frac{2}{\sqrt{2\pi}} e^{-\frac{v}{2}} \cdot \frac{1}{2\sqrt{v}} = \frac{1}{\sqrt{\pi}\sqrt{2}}\frac{1}{\sqrt{v}} e^{-\frac{v}{2}} = \frac{1}{\sqrt{\pi}\,2^{\frac{1}{2}}} v^{\frac{1}{2}-1} e^{-\frac{v}{2}}$$

Since $g(v)$ is a pdf ; i.e $\int_0^{\infty} \frac{1}{\sqrt{\pi}\,2^{\frac{1}{2}}} v^{\frac{1}{2}-1} e^{-\frac{v}{2}}\, dv = 1$ implies that $\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$.

Hence, $g(v) = \frac{1}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} v^{\frac{1}{2}-1} e^{-\frac{v}{2}} \; ; \; 0 < v < \infty$; i.e $V \sim \chi^2_{(r=1)}$.

Homework: 3.4 : 5, 8, 10, 12 and 13 pp. 175-176 .