Chapter 1

Probability and Distributions

1.1 Introduction

Many kinds of investigations may be characterized in part by the fact that repeated experimentation, under essentially the same conditions, is more or less standard procedure. For instance, in medical research, interest may center on the effect of a drug that is to be administered; or an economist may be concerned with the prices of three specified commodities at various time intervals; or the agronomist may wish to study the effect that a chemical fertilizer has on the yield of a cereal grain. The only way in which an investigator can elicit information about any such phenomenon is to perform the experiment. Each experiment terminates with an *outcome*. But it is characteristic of these experiments that the outcome cannot be predicted with certainty prior to the performance of the experiment.

Suppose that we have such an experiment, but the experiment is of such a nature that a collection of every possible outcome can be described prior to its performance. If this kind of experiment can be repeated under the same conditions, it is called a **random experiment**, and the collection of every possible outcome is called the experimental space or the **sample space**.

Example 1.1.1. In the toss of a coin, let the outcome tails be denoted by T and let the outcome heads be denoted by H. If we assume that the coin may be repeatedly tossed under the same conditions, then the toss of this coin is an example of a random experiment in which the outcome is one of the two symbols T and H; that is, the sample space is the collection of these two symbols. ∎

Example 1.1.2. In the cast of one red die and one white die, let the outcome be the ordered pair (number of spots up on the red die, number of spots up on the white die). If we assume that these two dice may be repeatedly cast under the same conditions, then the cast of this pair of dice is a random experiment. The sample space consists of the 36 ordered pairs: $(1,1), \ldots, (1,6), (2,1), \ldots, (2,6), \ldots, (6,6)$. ∎

Let $\mathcal{C}$ denote a sample space, let c denote an element of $\mathcal{C}$, and let C represent a collection of elements of $\mathcal{C}$. If, upon the performance of the experiment, the outcome is in C, we shall say that the *event* C has occurred. Now conceive of our having made N repeated performances of the random experiment. Then we can count the number f of times (the **frequency**) that the event C actually occurred throughout the N performances. The ratio f/N is called the **relative frequency** of the event C in these N experiments. A relative frequency is usually quite erratic for small values of N, as you can discover by tossing a coin. But as N increases, experience indicates that we associate with the event C a number, say p, that is equal or approximately equal to that number about which the relative frequency seems to stabilize. If we do this, then the number p can be interpreted as that number which, in future performances of the experiment, the relative frequency of the event C will either equal or approximate. Thus, although we *cannot* predict the outcome of a random experiment, we *can*, for a large value of N, predict approximately the relative frequency with which the outcome will be in C. The number p associated with the event C is given various names. Sometimes it is called the *probability* that the outcome of the random experiment is in C; sometimes it is called the *probability* of the event C; and sometimes it is called the *probability measure* of C. The context usually suggests an appropriate choice of terminology.

Example 1.1.3. Let $\mathcal{C}$ denote the sample space of Example 1.1.2 and let C be the collection of every ordered pair of $\mathcal{C}$ for which the sum of the pair is equal to seven. Thus C is the collection $(1,6), (2,5), (3,4), (4,3), (5,2)$, and $(6,1)$. Suppose that the dice are cast $N = 400$ times and let f, the frequency of a sum of seven, be $f = 60$. Then the relative frequency with which the outcome was in C is $f/N = \frac{60}{400} = 0.15$. Thus we might associate with C a number p that is close to 0.15, and p would be called the probability of the event C. ■

Remark 1.1.1. The preceding interpretation of probability is sometimes referred to as the *relative frequency approach*, and it obviously depends upon the fact that an experiment can be repeated under essentially identical conditions. However, many persons extend probability to other situations by treating it as a rational measure of belief. For example, the statement $p = \frac{2}{5}$ would mean to them that their *personal* or *subjective* probability of the event C is equal to $\frac{2}{5}$. Hence, if they are not opposed to gambling, this could be interpreted as a willingness on their part to bet on the outcome of C so that the two possible payoffs are in the ratio $p/(1-p) = \frac{2}{5}/\frac{3}{5} = \frac{2}{3}$. Moreover, if they truly believe that $p = \frac{2}{5}$ is correct, they would be willing to accept either side of the bet: (a) win 3 units if C occurs and lose 2 if it does not occur, or (b) win 2 units if C does not occur and lose 3 if it does. However, since the mathematical properties of probability given in Section 1.3 are consistent with either of these interpretations, the subsequent mathematical development does not depend upon which approach is used. ■

The primary purpose of having a mathematical theory of statistics is to provide mathematical models for random experiments. Once a model for such an experiment has been provided and the theory worked out in detail, the statistician may, within this framework, make inferences (that is, draw conclusions) about the random experiment. The construction of such a model requires a theory of probability. One of the more logically satisfying theories of probability is that based on the concepts of sets and functions of sets. These concepts are introduced in Section 1.2.

1.2 Set Theory

The concept of a *set* or a *collection* of objects is usually left undefined. However, a particular set can be described so that there is no misunderstanding as to what collection of objects is under consideration. For example, the set of the first 10 positive integers is sufficiently well described to make clear that the numbers $\frac{3}{4}$ and 14 are not in the set, while the number 3 is in the set. If an object belongs to a set, it is said to be an *element* of the set. For example, if C denotes the set of real numbers x for which $0 \leq x \leq 1$, then $\frac{3}{4}$ is an element of the set C. The fact that $\frac{3}{4}$ is an element of the set C is indicated by writing $\frac{3}{4} \in C$. More generally, $c \in C$ means that c is an element of the set C.

The sets that concern us are frequently *sets of numbers.* However, the language of sets of *points* proves somewhat more convenient than that of sets of numbers. Accordingly, we briefly indicate how we use this terminology. In analytic geometry considerable emphasis is placed on the fact that to each point on a line (on which an origin and a unit point have been selected) there corresponds one and only one number, say x; and that to each number x there corresponds one and only one point on the line. This one-to-one correspondence between the numbers and points on a line enables us to speak, without misunderstanding, of the "point x" instead of the "number x." Furthermore, with a plane rectangular coordinate system and with x and y numbers, to each symbol (x, y) there corresponds one and only one point in the plane; and to each point in the plane there corresponds but one such symbol. Here again, we may speak of the "point (x, y)," meaning the "ordered number pair x and y." This convenient language can be used when we have a rectangular coordinate system in a space of three or more dimensions. Thus the "point $(x_1, x_2, \ldots, x_n)$" means the numbers $x_1, x_2, \ldots, x_n$ in the order stated. Accordingly, in describing our sets, we frequently speak of a set of points (a set whose elements are points), being careful, of course, to describe the set so as to avoid any ambiguity. The notation $C = \{x : 0 \leq x \leq 1\}$ is read "C is the one-dimensional set of points x for which $0 \leq x \leq 1$." Similarly, $C = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ can be read "C is the two-dimensional set of points (x, y) that are interior to, or on the boundary of, a square with opposite vertices at $(0, 0)$ and $(1, 1)$."

We say a set C is **countable** if C is finite or has as many elements as there are positive integers. For example, the sets $C_1 = \{1, 2, \ldots, 100\}$ and $C_2 = \{1, 3, 5, 7, \ldots\}$ are countable sets. The interval of real numbers $(0, 1]$, though, is not countable.

We now give some definitions (together with illustrative examples) that lead to an elementary algebra of sets adequate for our purposes.

Definition 1.2.1. *If each element of a set C_1 is also an element of set C_2, the set C_1 is called a* **subset** *of the set C_2. This is indicated by writing $C_1 \subset C_2$. If $C_1 \subset C_2$ and also $C_2 \subset C_1$, the two sets have the same elements, and this is indicated by writing $C_1 = C_2$.*

Example 1.2.1. Let $C_1 = \{x : 0 \leq x \leq 1\}$ and $C_2 = \{x : -1 \leq x \leq 2\}$. Here the one-dimensional set C_1 is seen to be a subset of the one-dimensional set C_2; that is, $C_1 \subset C_2$. Subsequently, when the dimensionality of the set is clear, we do not make specific reference to it. ■

Example 1.2.2. Define the two sets $C_1 = \{(x, y) : 0 \leq x = y \leq 1\}$ and $C_2 = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Because the elements of C_1 are the points on one diagonal of the square, then $C_1 \subset C_2$. ■

Definition 1.2.2. *If a set C has no elements, C is called the* **null set**. *This is indicated by writing $C = \phi$.*

Definition 1.2.3. *The set of all elements that belong to at least one of the sets C_1 and C_2 is called the* **union** *of C_1 and C_2. The union of C_1 and C_2 is indicated by writing $C_1 \cup C_2$. The union of several sets $C_1, C_2, C_3, \ldots$ is the set of all elements that belong to at least one of the several sets, denoted by $C_1 \cup C_2 \cup C_3 \cup \cdots = \cup_{j=1}^{\infty} C_j$ or by $C_1 \cup C_2 \cup \cdots \cup C_k = \cup_{j=1}^{k} C_j$ if a finite number k of sets is involved.*

We refer to a union of the form $\cup_{j=1}^{\infty} C_j$ as a **countable union**.

Example 1.2.3. Define the sets $C_1 = \{x : x = 8, 9, 10, 11, \text{ or } 11 < x \leq 12\}$ and $C_2 = \{x : x = 0, 1, \ldots, 10\}$. Then

$$\begin{aligned} C_1 \cup C_2 &= \{x : x = 0, 1, \ldots, 8, 9, 10, 11, \text{ or } 11 < x \leq 12\} \\ &= \{x : x = 0, 1, \ldots, 8, 9, 10 \text{ or } 11 \leq x \leq 12\}. \end{aligned}$$ ■

Example 1.2.4. Define C_1 and C_2 as in Example 1.2.1. Then $C_1 \cup C_2 = C_2$. ■

Example 1.2.5. Let $C_2 = \phi$. Then $C_1 \cup C_2 = C_1$, for every set C_1. ■

Example 1.2.6. For every set C, $C \cup C = C$. ■

Example 1.2.7. Let

$$C_k = \left\{x : \tfrac{1}{k+1} \leq x \leq 1\right\}, \quad k = 1, 2, 3, \ldots.$$

Then $\cup_{k=1}^{\infty} C_k = \{x : 0 < x \leq 1\}$. Note that the number zero is not in this set, since it is not in one of the sets $C_1, C_2, C_3, \ldots$. ■

Definition 1.2.4. *The set of all elements that belong to each of the sets C_1 and C_2 is called the* **intersection** *of C_1 and C_2. The intersection of C_1 and C_2 is indicated by writing $C_1 \cap C_2$. The intersection of several sets $C_1, C_2, C_3, \ldots$ is the set of all elements that belong to each of the sets $C_1, C_2, C_3, \ldots$. This intersection is denoted by $C_1 \cap C_2 \cap C_3 \cap \cdots = \cap_{j=1}^{\infty} C_j$ or by $C_1 \cap C_2 \cap \cdots \cap C_k = \cap_{j=1}^{k} C_j$ if a finite number k of sets is involved.*

We refer to an intersection of the form $\cap_{j=1}^{\infty} C_j$ as a **countable intersection**.

Example 1.2.8. Let $C_1 = \{(0,0), (0,1), (1,1)\}$ and $C_2 = \{(1,1), (1,2), (2,1)\}$. Then $C_1 \cap C_2 = \{(1,1)\}$. ■

Example 1.2.9. Let $C_1 = \{(x, y) : 0 \leq x + y \leq 1\}$ and $C_2 = \{(x, y) : 1 < x + y\}$. Then C_1 and C_2 have no points in common and $C_1 \cap C_2 = \phi$. ■

Example 1.2.10. For every set C, $C \cap C = C$ and $C \cap \phi = \phi$. ■

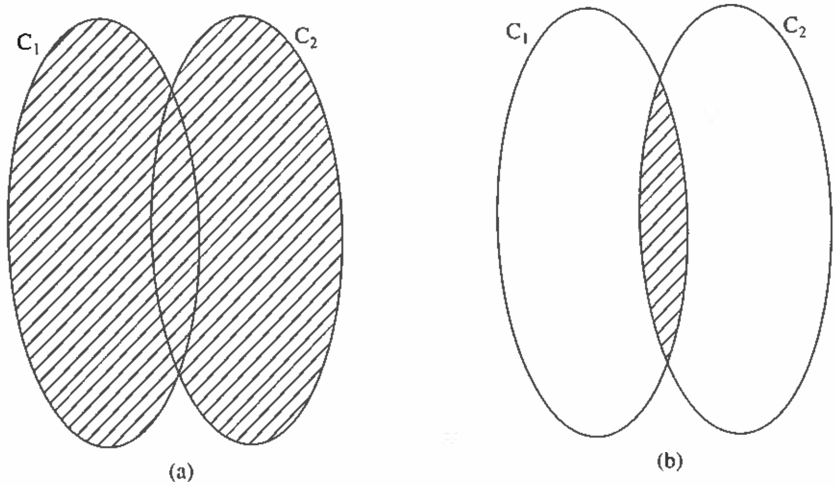

Figure 1.2.1: (a) $C_1 \cup C_2$ and (b) $C_1 \cap C_2$.

Example 1.2.11. Let

$$C_k = \{x : 0 < x < \tfrac{1}{k}\}, \quad k = 1, 2, 3, \ldots$$

Then $\cap_{k=1}^{\infty} C_k = \phi$, because there is no point that belongs to each of the sets $C_1, C_2, C_3, \ldots$. ■

Example 1.2.12. Let C_1 and C_2 represent the sets of points enclosed, respectively, by two intersecting ellipses. Then the sets $C_1 \cup C_2$ and $C_1 \cap C_2$ are represented, respectively, by the shaded regions in the **Venn diagrams** in Figure 1.2.1. ■

Definition 1.2.5. *In certain discussions or considerations, the totality of all elements that pertain to the discussion can be described. This set of all elements under consideration is given a special name. It is called the* **space**. *We often denote spaces by letters such as $\mathcal{C}$ and $\mathcal{D}$.*

Example 1.2.13. Let the number of heads, in tossing a coin four times, be denoted by x. Of necessity, the number of heads is of the numbers $0, 1, 2, 3, 4$. Here, then, the space is the set $\mathcal{C} = \{0, 1, 2, 3, 4\}$. ■

Example 1.2.14. Consider all nondegenerate rectangles of base x and height y. To be meaningful, both x and y must be positive. Then the space is given by the set $\mathcal{C} = \{(x, y) : x > 0, y > 0\}$. ■

Definition 1.2.6. *Let $\mathcal{C}$ denote a space and let C be a subset of the set $\mathcal{C}$. The set that consists of all elements of $\mathcal{C}$ that are not elements of C is called the* **complement** *of C (actually, with respect to $\mathcal{C}$). The complement of C is denoted by C^c. In particular, $\mathcal{C}^c = \phi$.*

Example 1.2.15. Let $\mathcal{C}$ be defined as in Example 1.2.13, and let the set $C = \{0, 1\}$. The complement of C (with respect to $\mathcal{C}$) is $C^c = \{2, 3, 4\}$. ■

Example 1.2.16. Given $C \subset \mathcal{C}$. Then $C \cup C^c = \mathcal{C}, C \cap C^c = \phi$, $C \cup \mathcal{C} = \mathcal{C}$, $C \cap \mathcal{C} = C$, and $(C^c)^c = C$. ■

Example 1.2.17 (DeMorgan's Laws). A set of useful rules is known as DeMorgan's Laws. Let $\mathcal{C}$ denote a space and let $C_i \subset \mathcal{C}$, $i = 1, 2$. Then

$$(C_1 \cap C_2)^c = C_1^c \cup C_2^c \tag{1.2.1}$$

$$(C_1 \cup C_2)^c = C_1^c \cap C_2^c. \tag{1.2.2}$$

The reader is asked to prove these in Exercise 1.2.4 and to extend them to countable unions and intersections. ■

Many of the functions used in calculus and in this book are functions which map real numbers into real numbers. We are often, however, concerned with functions that map sets into real numbers. Such functions are naturally called functions of a set or, more simply, **set functions**. Next we give some examples of set functions and evaluate them for certain simple sets.

Example 1.2.18. Let C be a set in one-dimensional space and let $Q(C)$ be equal to the number of points in C which correspond to positive integers. Then $Q(C)$ is a function of the set C. Thus, if $C = \{x : 0 < x < 5\}$, then $Q(C) = 4$; if $C = \{-2, -1\}$, then $Q(C) = 0$; if $C = \{x : -\infty < x < 6\}$, then $Q(C) = 5$. ■

Example 1.2.19. Let C be a set in two-dimensional space and let $Q(C)$ be the area of C if C has a finite area; otherwise, let $Q(C)$ be undefined. Thus, if $C = \{(x, y) : x^2 + y^2 \le 1\}$, then $Q(C) = \pi$; if $C = \{(0, 0), (1, 1), (0, 1)\}$, then $Q(C) = 0$; if $C = \{(x, y) : 0 \le x, 0 \le y, x + y \le 1\}$, then $Q(C) = \frac{1}{2}$. ■

Example 1.2.20. Let C be a set in three-dimensional space and let $Q(C)$ be the volume of C if C has a finite volume; otherwise, let $Q(C)$ be undefined. Thus, if $C = \{(x, y, z) : 0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3\}$, then $Q(C) = 6$; if $C = \{(x, y, z) : x^2 + y^2 + z^2 \ge 1\}$, then $Q(C)$ is undefined. ■

At this point we introduce the following notations. The symbol

$$\int_C f(x)\, dx$$

means the ordinary (Riemann) integral of $f(x)$ over a prescribed one-dimensional set C; the symbol

$$\int\int_C g(x, y)\, dxdy$$

means the Riemann integral of $g(x, y)$ over a prescribed two-dimensional set C; and so on. To be sure, unless these sets C and these functions $f(x)$ and $g(x, y)$ are chosen with care, the integrals frequently fail to exist. Similarly, the symbol

$$\sum_C f(x)$$

means the sum extended over all $x \in C$; the symbol

$$\sum_C \sum g(x, y)$$

means the sum extended over all $(x, y) \in C$; and so on.

Example 1.2.21. Let C be a set in one-dimensional space and let $Q(C) = \sum_C f(x)$, where

$$f(x) = \begin{cases} (\frac{1}{2})^x & x = 1, 2, 3, \ldots \\ 0 & \text{elsewhere.} \end{cases}$$

If $C = \{x : 0 \le x \le 3\}$, then

$$Q(C) = \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 = \tfrac{7}{8}. \ \blacksquare$$

Example 1.2.22. Let $Q(C) = \sum_C f(x)$, where

$$f(x) = \begin{cases} p^x(1-p)^{1-x} & x = 0, 1 \\ 0 & \text{elsewhere.} \end{cases}$$

If $C = \{0\}$, then

$$Q(C) = \sum_{x=0}^{0} p^x(1-p)^{1-x} = 1 - p;$$

if $C = \{x : 1 \le x \le 2\}$, then $Q(C) = f(1) = p$. $\blacksquare$

Example 1.2.23. Let C be a one-dimensional set and let

$$Q(C) = \int_C e^{-x}\,dx.$$

Thus, if $C = \{x : 0 \le x < \infty\}$, then

$$Q(C) = \int_0^\infty e^{-x}\,dx = 1;$$

if $C = \{x : 1 \le x \le 2\}$, then

$$Q(C) = \int_1^2 e^{-x}\,dx = e^{-1} - e^{-2};$$

if $C_1 = \{x : 0 \le x \le 1\}$ and $C_2 = \{x : 1 < x \le 3\}$, then

$$\begin{aligned} Q(C_1 \cup C_2) &= \int_0^3 e^{-x}\,dx \\ &= \int_0^1 e^{-x}\,dx + \int_1^3 e^{-x}\,dx \\ &= Q(C_1) + Q(C_2). \ \blacksquare \end{aligned}$$

Example 1.2.24. Let C be a set in n-dimensional space and let

$$Q(C) = \int_C \cdots \int dx_1 dx_2 \cdots dx_n.$$

If $C = \{(x_1, x_2, \ldots, x_n) : 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\}$, then

$$\begin{aligned} Q(C) &= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} \int_0^{x_2} dx_1 dx_2 \cdots dx_{n-1} dx_n \\ &= \frac{1}{n!}, \end{aligned}$$

where $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$. ■

EXERCISES

1.2.1. Find the union $C_1 \cup C_2$ and the intersection $C_1 \cap C_2$ of the two sets C_1 and C_2, where

(a) $C_1 = \{0, 1, 2, \}$, $C_2 = \{2, 3, 4\}$.

(b) $C_1 = \{x : 0 < x < 2\}$, $C_2 = \{x : 1 \le x < 3\}$.

(c) $C_1 = \{(x, y) : 0 < x < 2, 1 < y < 2\}$, $C_2 = \{(x, y) : 1 < x < 3, 1 < y < 3\}$.

1.2.2. Find the complement C^c of the set C with respect to the space $\mathcal{C}$ if

(a) $\mathcal{C} = \{x : 0 < x < 1\}$, $C = \{x : \frac{5}{8} < x < 1\}$.

(b) $\mathcal{C} = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$, $C = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

(c) $\mathcal{C} = \{(x, y) : |x| + |y| \le 2\}$, $C = \{(x, y) : x^2 + y^2 < 2\}$.

1.2.3. List all possible arrangements of the four letters m, a, r, and y. Let C_1 be the collection of the arrangements in which y is in the last position. Let C_2 be the collection of the arrangements in which m is in the first position. Find the union and the intersection of C_1 and C_2.

1.2.4. Referring to Example 1.2.17, verify DeMorgan's Laws (1.2.1) and (1.2.2) by using Venn diagrams and then prove that the laws are true. Generalize the laws to countable unions and intersections.

1.2.5. By the use of Venn diagrams, in which the space $\mathcal{C}$ is the set of points enclosed by a rectangle containing the circles C_1, C_2, and C_3, compare the following sets. These laws are called the **distributive laws**.

(a) $C_1 \cap (C_2 \cup C_3)$ and $(C_1 \cap C_2) \cup (C_1 \cap C_3)$.

(b) $C_1 \cup (C_2 \cap C_3)$ and $(C_1 \cup C_2) \cap (C_1 \cup C_3)$.

1.2.6. If a sequence of sets $C_1, C_2, C_3, \ldots$ is such that $C_k \subset C_{k+1}$, $k = 1, 2, 3, \ldots$, the sequence is said to be a *nondecreasing sequence*. Give an example of this kind of sequence of sets.

1.2.7. If a sequence of sets $C_1, C_2, C_3, \ldots$ is such that $C_k \supset C_{k+1}$, $k = 1, 2, 3, \ldots$, the sequence is said to be a *nonincreasing sequence*. Give an example of this kind of sequence of sets.

1.2.8. Suppose $C_1, C_2, C_3, \ldots$ is a *nondecreasing sequence* of sets, i.e., $C_k \subset C_{k+1}$, for $k = 1, 2, 3, \ldots$. Then $\lim_{k\to\infty} C_k$ is defined as the union $C_1 \cup C_2 \cup C_3 \cup \cdots$. Find $\lim_{k\to\infty} C_k$ if

(a) $C_k = \{x : 1/k \le x \le 3 - 1/k\}$, $k = 1, 2, 3, \ldots$.

(b) $C_k = \{(x, y) : 1/k \le x^2 + y^2 \le 4 - 1/k\}$, $k = 1, 2, 3, \ldots$.

1.2.9. If $C_1, C_2, C_3, \ldots$ are sets such that $C_k \supset C_{k+1}$, $k = 1, 2, 3, \ldots$, $\lim_{k\to\infty} C_k$ is defined as the intersection $C_1 \cap C_2 \cap C_3 \cap \cdots$. Find $\lim_{k\to\infty} C_k$ if

(a) $C_k = \{x : 2 - 1/k < x \le 2\}$, $k = 1, 2, 3, \ldots$.

(b) $C_k = \{x : 2 < x \le 2 + 1/k\}$, $k = 1, 2, 3, \ldots$.

(c) $C_k = \{(x, y) : 0 \le x^2 + y^2 \le 1/k\}$, $k = 1, 2, 3, \ldots$.

1.2.10. For every one-dimensional set C, define the function $Q(C) = \sum_C f(x)$, where $f(x) = (\frac{2}{3})(\frac{1}{3})^x$, $x = 0, 1, 2, \ldots$, zero elsewhere. If $C_1 = \{x : x = 0, 1, 2, 3\}$ and $C_2 = \{x : x = 0, 1, 2, \ldots\}$, find $Q(C_1)$ and $Q(C_2)$.
Hint: Recall that $S_n = a + ar + \cdots + ar^{n-1} = a(1 - r^n)/(1 - r)$ and, hence, it follows that $\lim_{n\to\infty} S_n = a/(1 - r)$ provided that $|r| < 1$.

1.2.11. For every one-dimensional set C for which the integral exists, let $Q(C) = \int_C f(x)\,dx$, where $f(x) = 6x(1 - x)$, $0 < x < 1$, zero elsewhere; otherwise, let $Q(C)$ be undefined. If $C_1 = \{x : \frac{1}{4} < x < \frac{3}{4}\}$, $C_2 = \{\frac{1}{2}\}$, and $C_3 = \{x : 0 < x < 10\}$, find $Q(C_1)$, $Q(C_2)$, and $Q(C_3)$.

1.2.12. For every two-dimensional set C contained in R^2 for which the integral exists, let $Q(C) = \int\int_C (x^2 + y^2)\,dxdy$. If $C_1 = \{(x, y) : -1 \le x \le 1, -1 \le y \le 1\}$, $C_2 = \{(x, y) : -1 \le x = y \le 1\}$, and $C_3 = \{(x, y) : x^2 + y^2 \le 1\}$, find $Q(C_1)$, $Q(C_2)$, and $Q(C_3)$.

1.2.13. Let $\mathcal{C}$ denote the set of points that are interior to, or on the boundary of, a square with opposite vertices at the points $(0, 0)$ and $(1, 1)$. Let $Q(C) = \int\int_C dy\,dx$.

(a) If $C \subset \mathcal{C}$ is the set $\{(x, y) : 0 < x < y < 1\}$, compute $Q(C)$.

(b) If $C \subset \mathcal{C}$ is the set $\{(x, y) : 0 < x = y < 1\}$, compute $Q(C)$.

(c) If $C \subset \mathcal{C}$ is the set $\{(x, y) : 0 < x/2 \le y \le 3x/2 < 1\}$, compute $Q(C)$.

1.2.14. Let $\mathcal{C}$ be the set of points interior to or on the boundary of a cube with edge of length 1. Moreover, say that the cube is in the first octant with one vertex at the point $(0, 0, 0)$ and an opposite vertex at the point $(1, 1, 1)$. Let $Q(C) = \int\int\int_C dxdydz$.

(a) If $C \subset \mathcal{C}$ is the set $\{(x, y, z) : 0 < x < y < z < 1\}$, compute $Q(C)$.