Section 1.2 Set Theory

Element of the set: If an element belongs to a set, it's said to be an element of the set. Generally **a** is an element of the set **A**, $a \in A$ **.**

Example 1: Let **C** be the set of the first five positive integers.
$C = \{x; 1, 2, 3, 4, 5\}$ Then $1 \in C$ but $\frac{1}{2} \notin C$.

Example 2: Let **C** be the set of real numbers x for which $-1 \le x \le 1$.
$C = \{x;\ -1 \le x \le 1\}$ Then **C** is a one dimensional set.
Similarly, if is $C = \{(x, y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$
then **C** is a two dimensional set.

Definition 1.2.1: If every element of a set C_1 is also an element of the C_2 ,
then the set C_1 is called a subset of the set C_2 . $C_1 \subset C_2$
If $C_1 \subset C_2$ and $C_2 \subset C_1$ then $C_1 = C_2$.

Example 3: Let $C_1 = \{x;\ 0, 1, 2\}$ and $C_2 = \{x;\ -1 \le x \le 2\}$ then $C_1 \subset C_2$.
Let $C_1 = \{x;\ 0, 1, 2\}$ and $C_2 = \{x; -1 \le x < 2\}$ then $C_1 \not\subset C_2$.

Example 1.2.2 p4: Let $C_1 = \{(x, y);\ 0 \le x = y \le 1\}$ and
$C_2 = \{(x, y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$ then $C_1 \subset C_2$.

Definition 1.2.2: If a set C has no element, C is called the null set. $C = \varnothing$.

Definition 1.2.3: The set of all elements that belong to at least one of the sets C_1 and C_2 is called the union of C_1 and C_2 . $C_1 \cup C_2$
For **k** sets $C_1, C_2, C_3, \ldots, C_{k-1}$, C_k the union is
$C_1 \cup C_2 \cup C_3 \cup, \ldots, \cup C_{k-1} \cup C_k$.

Do examples 1.2.3 through 1.2.7 page 4

Example 1.2.3. Define the sets $C_1 = \{x : x = 8, 9, 10, 11, \text{ or } 11 < x \leq 12\}$ and $C_2 = \{x : x = 0, 1, \ldots, 10\}$. Then

$$\begin{aligned} C_1 \cup C_2 &= \{x : x = 0, 1, \ldots, 8, 9, 10, 11, \text{ or } 11 < x \leq 12\} \\ &= \{x : x = 0, 1, \ldots, 8, 9, 10 \text{ or } 11 \leq x \leq 12\}. \blacksquare \end{aligned}$$

Example 1.2.4. Define C_1 and C_2 as in Example 1.2.1. Then $C_1 \cup C_2 = C_2$. ■

Example 1.2.5. Let $C_2 = \phi$. Then $C_1 \cup C_2 = C_1$, for every set C_1. ■

Example 1.2.6. For every set C, $C \cup C = C$. ■

Example 1.2.7. Let

$$C_k = \left\{x : \tfrac{1}{k+1} \leq x \leq 1\right\}, \quad k = 1, 2, 3, \ldots.$$

Then $\cup_{k=1}^{\infty} C_k = \{x : 0 < x \leq 1\}$. Note that the number zero is not in this set, since it is not in one of the sets $C_1, C_2, C_3, \ldots$. ■

Definition 1.2.4: The set of all elements that belong to each of the sets C_1 and C_2 is called the intersection of C_1 and C_2 . $C_1 \cap C_2$

For **k** sets $C_1, C_2, C_3, \ldots, C_{k-1}, C_k$ the intersection is $C_1 \cap C_2 \cap C_3 \cap, \ldots, \cap C_{k-1} \cap C_k$.

Do examples 1.2.8 through 1.2.12 pages 4-5

Example 1.2.8. Let $C_1 = \{(0,0),(0,1),(1,1)\}$ and $C_2 = \{(1,1),(1,2),(2,1)\}$. Then $C_1 \cap C_2 = \{(1,1)\}$. ■

Example 1.2.9. Let $C_1 = \{(x,y) : 0 \le x + y \le 1\}$ and $C_2 = \{(x,y) : 1 < x + y\}$. Then C_1 and C_2 have no points in common and $C_1 \cap C_2 = \phi$. ■

Example 1.2.10. For every set C, $C \cap C = C$ and $C \cap \phi = \phi$. ■

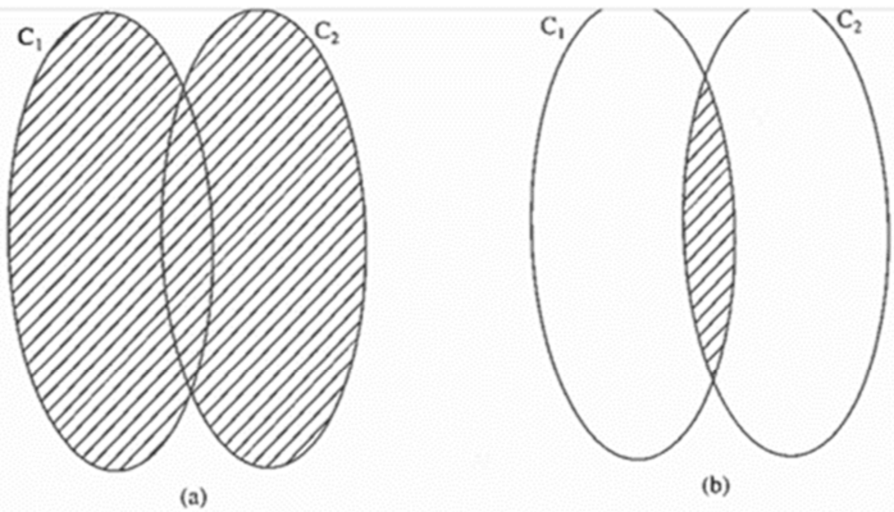


Figure 1.2.1: (a) $C_1 \cup C_2$ and (b) $C_1 \cap C_2$.

Example 1.2.11. Let

$$C_k = \{x : 0 < x < \tfrac{1}{k}\}, \quad k = 1, 2, 3, \ldots$$

Then $\cap_{k=1}^{\infty} C_k = \phi$, because there is no point that belongs to each of the sets $C_1, C_2, C_3, \ldots$. ■

Example 1.2.12. Let C_1 and C_2 represent the sets of points enclosed, respectively, by two intersecting ellipses. Then the sets $C_1 \cup C_2$ and $C_1 \cap C_2$ are represented, respectively, by the shaded regions in the **Venn diagrams** in Figure 1.2.1. ■

Definition 1.2.6: The set that consists of all elements of $\mathscr{C}$ that are not elements C is called the complement of C. C^c and $\mathscr{C}^c = \varnothing$.

Do examples 1.2.16 and 1.2.17 **page 6**

Example 1.2.15. Let $\mathcal{C}$ be defined as in Example 1.2.13, and let the set $C = \{0, 1\}$. The complement of C (with respect to $\mathcal{C}$) is $C^c = \{2, 3, 4\}$. ■

Example 1.2.16. Given $C \subset \mathcal{C}$. Then $C \cup C^c = \mathcal{C}, C \cap C^c = \phi$, $C \cup \mathcal{C} = \mathcal{C}$, $C \cap \mathcal{C} = C$, and $(C^c)^c = C$. ■

Example 1.2.17 (DeMorgan's Laws). A set of useful rules is known as DeMorgan's Laws. Let $\mathcal{C}$ denote a space and let $C_i \subset \mathcal{C}$, $i = 1, 2$. Then

$$(C_1 \cap C_2)^c = C_1^c \cup C_2^c \quad (1.2.1)$$

$$(C_1 \cup C_2)^c = C_1^c \cap C_2^c. \quad (1.2.2)$$

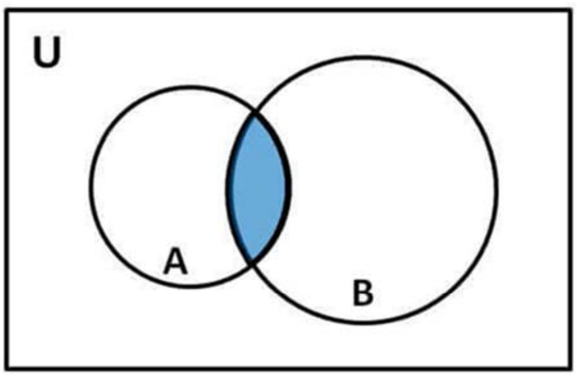
Venn Diagrams . Do example 1.2.12 on page 5. Also, can you use Venn diagrams to demonstrate DeMorgan's Law on example 1.2.17 on page 6 (see HW)?

Homework: **1, 2 (a, b, c) , 3, 4 (Verify Only Using Venn Diagrams, Do Not Do Proof), and 5 on page 8.**
Note: For 2c find P for C and C^c**.**

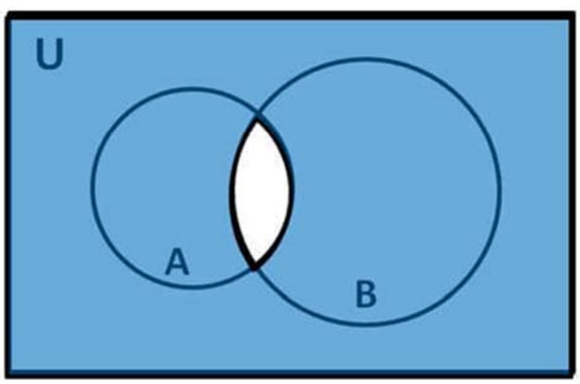
De Morgan's Law

Proving $(A \cap B)' = A' \cup B'$

$(A \cap B)'$

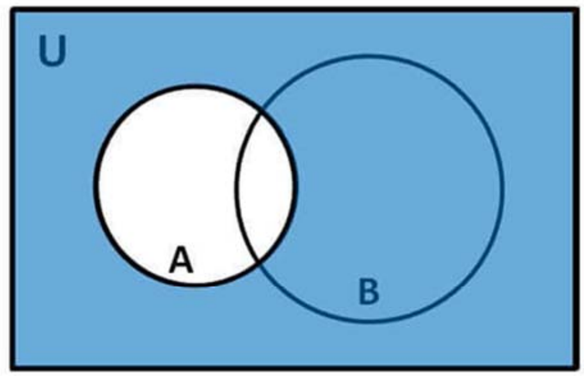


$A \cap B$

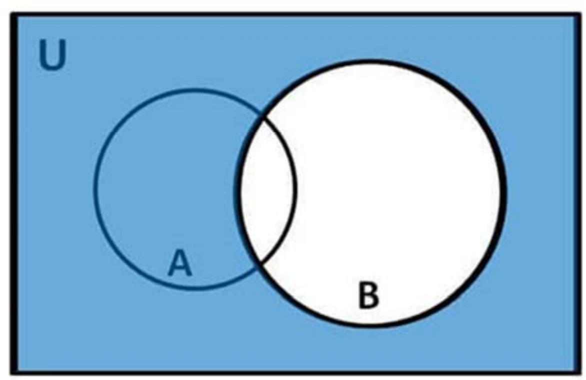


$(A \cap B)'$

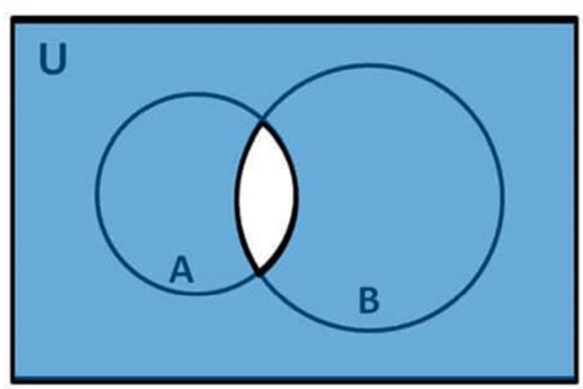
$A' \cup B'$



A'



B'



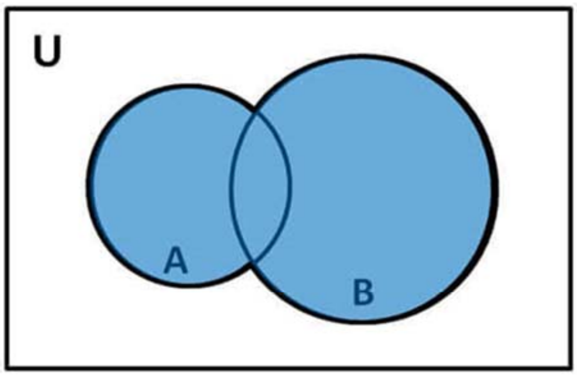
$A' \cup B'$

$\therefore (A \cap B)' = A' \cup B'$

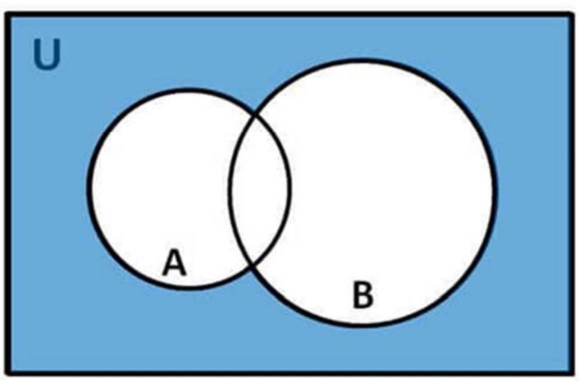
De Morgan's Law

Proving $(A \cup B)' = A' \cap B'$

$(A \cup B)'$

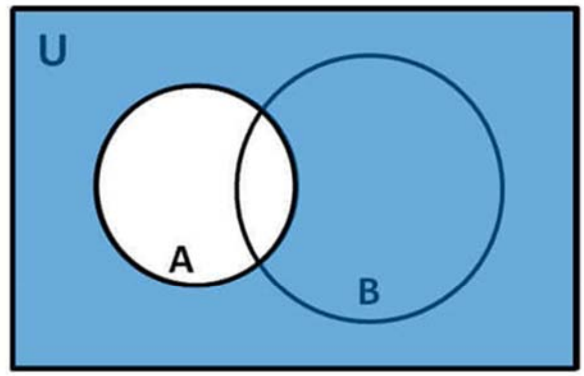


$A \cup B$

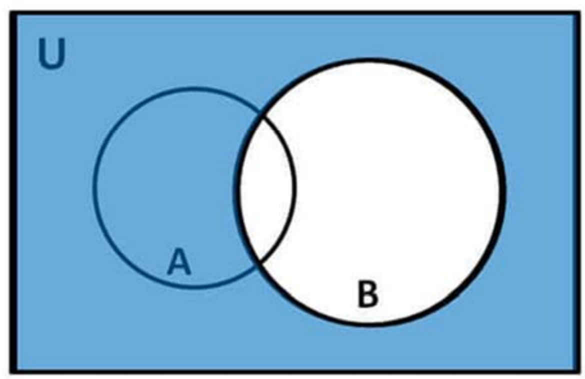


$(A \cup B)'$

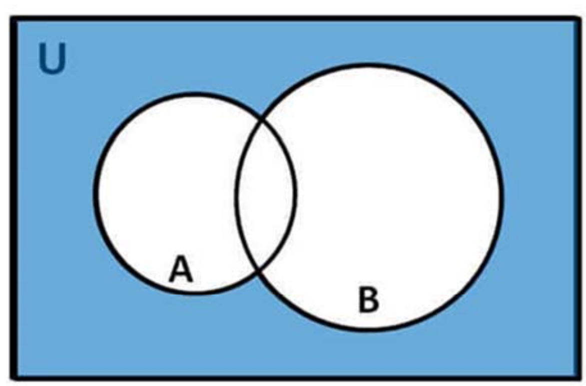
$A' \cap B'$



A'



B'



$A' \cap B'$

$\therefore (A \cup B)' = A' \cap B'$

Section 1.2 **Continues-Set Functions**

Set Function: If the function is evaluated over an entire set of points is called a set function. Functions that are evaluated at one point are called point functions.

Example 1.2.18 p6: Let C be a one dimensional set and let $Q(C)$ be equal to the number of points in C which corresponds to positive integers. Then $Q(C)$ is a function of the set C. If $C=\{x;\ 0<x<5\}$ then $Q(C)=4$. If $C=\{x;\ -2,-1\}$ then $Q(C)=0$. If $C=\{x;\ -\infty<x<6\}$ then $Q(C)=5$.

Do example 1.2.19 page 6

Example 1.2.19. Let C be a set in two-dimensional space and let $Q(C)$ be the area of C if C has a finite area; otherwise, let $Q(C)$ be undefined. Thus, if $C = \{(x,y) : x^2+y^2 \le 1\}$, then $Q(C)=\pi$; if $C=\{(0,0),(1,1),(0,1)\}$, then $Q(C)=0$; if $C=\{(x,y): 0\le x, 0\le y, x+y\le 1\}$, then $Q(C)=\frac{1}{2}$. ■

Notation: The symbol $\int_c f(x)\,dx$ will mean the ordinary (Riemann) integral of $f(x)$ over a prescribed one dimensional set. $\iint_c g(x,y)\,dxdy$ Two dimensional set.

Similarly, $\sum_c f(x)$ means the sum extended over all $x\in C$. $\sum\sum_c g(x,y)$ means the sum extended over all $(x,y)\in C$.

Do examples 1.2.21, 1.2.22, 1.2.23, and 1.2.24 pages 7-8 .

Example 1.2.21. Let C be a set in one-dimensional space and let $Q(C)=\sum_C f(x)$, where

$$f(x)=\begin{cases} (\frac{1}{2})^x & x=1,2,3,\dots \\ 0 & \text{elsewhere.}\end{cases}$$

If $C=\{x: 0\le x\le 3\}$, then

$$Q(C)=\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3=\frac{7}{8}. \ ■$$

Example 1.2.22. Let $Q(C)=\sum_C f(x)$, where

$$f(x)=\begin{cases} p^x(1-p)^{1-x} & x=0,1 \\ 0 & \text{elsewhere.}\end{cases}$$

If $C=\{0\}$, then

$$Q(C)=\sum_{x=0}^{0} p^x(1-p)^{1-x}=1-p;$$

if $C=\{x: 1\le x\le 2\}$, then $Q(C)=f(1)=p$. ■

Example 1.2.23. Let C be a one-dimensional set and let

$$Q(C) = \int_C e^{-x}dx.$$

Thus, if $C = \{x : 0 \le x < \infty\}$, then

$$Q(C) = \int_0^\infty e^{-x}dx = 1;$$

if $C = \{x : 1 \le x \le 2\}$, then

$$Q(C) = \int_1^2 e^{-x}dx = e^{-1} - e^{-2};$$

if $C_1 = \{x : 0 \le x \le 1\}$ and $C_2 = \{x : 1 < x \le 3\}$, then

$$\begin{aligned} Q(C_1 \cup C_2) &= \int_0^3 e^{-x}dx \\ &= \int_0^1 e^{-x}dx + \int_1^3 e^{-x}dx \\ &= Q(C_1) + Q(C_2). \ \blacksquare \end{aligned}$$

Example 1.2.24. Let C be a set in n-dimensional space and let

$$Q(C) = \int \cdots \int_C dx_1 dx_2 \cdots dx_n.$$

If $C = \{(x_1, x_2, \ldots, x_n) : 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\}$, then

$$\begin{aligned} Q(C) &= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} \int_0^{x_2} dx_1 dx_2 \cdots dx_{n-1} dx_n \\ &= \frac{1}{n!}, \end{aligned}$$

where $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$. $\blacksquare$

Homework: 10, 11 , 12, 13, and 15 on pages 9-10 .

ERTING COORDINATES

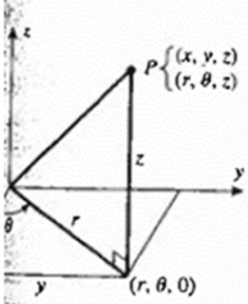
Just as we needed to convert between rectangular and polar coordinates in 2-space, so we will need to be able to convert between rectangular, cylindrical, and spherical coordinates in 3-space. Table 13.8.1 provides formulas for making these conversions.

Table 13.8.1

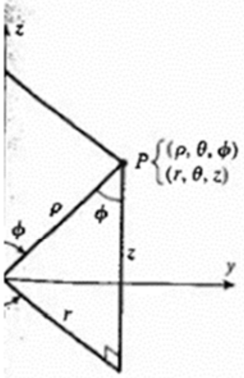
CONVERSION		FORMULAS	RESTRICTIONS
Cylindrical to rectangular	$(r, \theta, z) \to (x, y, z)$	$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$	
Rectangular to cylindrical	$(x, y, z) \to (r, \theta, z)$	$r = \sqrt{x^2+y^2},\ \tan\theta = y/x,\ z = z$	
Spherical to cylindrical	$(\rho, \theta, \phi) \to (r, \theta, z)$	$r = \rho\sin\phi,\quad \theta = \theta,\ z = \rho\cos\phi$	$r \ge 0, \rho \ge 0$
Cylindrical to spherical	$(r, \theta, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{r^2+z^2},\ \theta = \theta,\ \tan\phi = r/z$	$0 \le \theta < 2\pi$
Spherical to rectangular	$(\rho, \theta, \phi) \to (x, y, z)$	$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$	$0 \le \phi \le \pi$
Rectangular to spherical	$(x, y, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{x^2+y^2+z^2},\ \tan\theta = y/x,\ \cos\phi = z/\sqrt{x^2+y^2+z^2}$	

$|J| = r$

$|J| = \rho^2 \sin\phi$



(a)



(b)

.8.3

The diagrams in Figure 13.8.3 will help you to understand how the formulas in Table 13.8.1 are derived. For example, part (*a*) of the figure shows that in converting between rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z), we can interpret (r, θ) as polar coordinates of (x, y). Thus, the polar-to-rectangular and rectangular-to-polar conversion formulas (1) and (2) of Section 12.1 provide the conversion formulas between rectangular and cylindrical coordinates in the table.

Part (*b*) of Figure 13.8.3 suggests that the spherical coordinates (ρ, θ, ϕ) of a point P can be converted to cylindrical coordinates (r, θ, z) by the conversion formulas

$$r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi \tag{1}$$

Moreover, since the cylindrical coordinates (r, θ, z) of P can be converted to rectangular coordinates (x, y, z) by the conversion formulas

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{2}$$

we can obtain direct conversion formulas from spherical coordinates to rectangular coordinates by substituting (1) in (2). This yields

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi \tag{3}$$

The other conversion formulas in Table 13.8.1 are left as exercises.

Example 1

(a) Find the rectangular coordinates of the point with cylindrical coordinates

$$(r, \theta, z) = (4, \pi/3, -3)$$

(b) Find the rectangular coordinates of the point with spherical coordinates

$$(\rho, \theta, \phi) = (4, \pi/3, \pi/4)$$