Chapter 5
Probability and Random Variables

Section 5.1 Basic Concepts of Probability

Experiment: An act or process that generates well-defined outcomes.

Example:

1. Toss a coin.
2. Roll a die.
3. Selecting a random sample of size 2 from a group of five.

Sample Space: The collection of all possible outcomes of an experiment.

Simple Event: An individual outcome to an experiment.

Event: a subset (part) of the sample space.

Now we wish to assign probabilities to experimental outcomes. There are two approaches that are used most frequently.

1. The Classical Approach
2. Empirical or The Relative Frequency Approach

Regardless of the method used, the probabilities assigned must satisfy two basic requirements:

1. The probability assigned to each experimental outcome E must be between 0 and 1. That is,

$$0 \le P(E) \le 1 \ .$$

2. If $S = \{e_1, e_2, \ldots, e_n\}$, then $P(e_1) + P(e_2) + \ldots + P(e_n) = P(S) = 1$. If all the probabilities are in the sample space are between 0 and 1 and all of them sum to 1 then we have a **PROBABILITY MODEL.**

<u>**Note:**</u> The probability of an event is the sum of the probabilities of the simple events which comprise it.

A Probability Model

The color of a plain M&M milk chocolate candy can be brown, yellow, red, blue, orange, or green. Suppose a candy is randomly selected from a bag. Table 1 shows each color and the probability of drawing that color.

To verify that this is a probability model, we must show that Rules 1 and 2 are satisfied.

Each probability is greater than or equal to 0 and less than or equal to 1, so Rule 1 is satisfied.

Because

$$0.13 + 0.14 + 0.13 + 0.24 + 0.20 + 0.16 = 1$$

Rule 2 is also satisfied. The table is an example of a probability model. •

Table 1

Color	Probability
Brown	0.13
Yellow	0.14
Red	0.13
Blue	0.24
Orange	0.20
Green	0.16

If an event is **impossible**, the probability of the event is 0. If an event is a **certainty**, the probability of the event is 1.

The closer a probability is to 1, the more likely the event will occur. The closer a probability is to 0, the less likely the event will occur. For example, an event with probability 0.8 is more likely to occur than an event with probability 0.75. An event with probability 0.8 will occur about 80 times out of 100 repetitions of the experiment, whereas an event with probability 0.75 will occur about 75 times out of 100.

Be careful of this interpretation. An event with a probability of 0.75 does not have to occur 75 times out of 100. Rather, we *expect* the number of occurrences to be close to 75 in 100 trials. The more repetitions of the probability experiment, the closer the proportion with which the event occurs will be to 0.75 (the Law of Large Numbers).

One goal of this course is to learn how probabilities can be used to identify ***unusual events***.

An **unusual event** is an event that has a low probability of occurring.

Typically, an event with a probability less than 0.05 (or 5%) is considered unusual.

We have three different methods of assigning probabilities, **Subjective**, **Classical**, and **Empirical**.

Subjective probability is a type of probability derived from an individual's personal judgment. For example, what is the probability that the VSU football team will make it to the Championship game? I say 60%, you say 80%. Both of our answers are based on personal judgment.

Classical Method: Assume that a given experiment has n different outcomes, each of which has an equal chance of occurring. Then,

Probability of each outcome = P(each outcome) = $\frac{1}{n}$.

Example: Roll a pair of fair dice

Computing Probabilities Using the Classical Method

Problem A pair of fair dice is rolled. Fair die are die where each outcome is equally likely.

(a) Compute the probability of rolling a seven.

(b) Compute the probability of rolling "snake eyes"; that is, compute the probability of rolling a two.

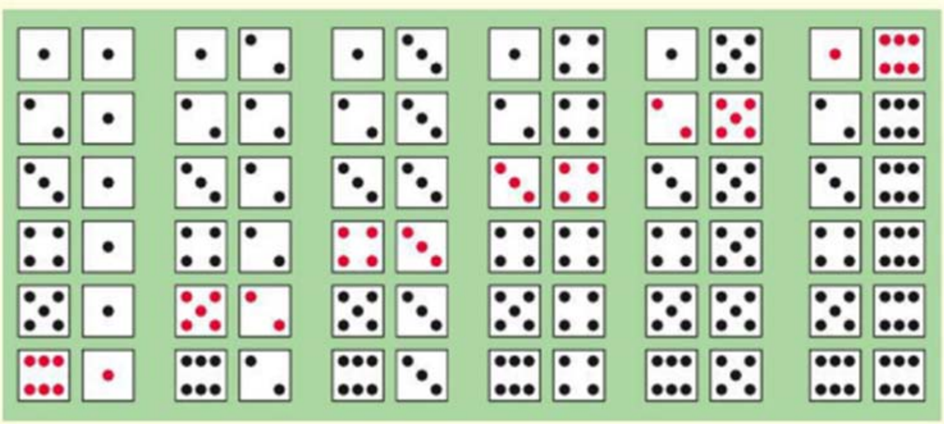
(c) Comment on the likelihood of rolling a seven versus rolling a two.

Approach To compute probabilities using the classical method, count the number of outcomes in the sample space and count the number of ways the event can occur. Then, divide the number of outcomes by the number of ways the event can occur.

Solution

(a) There are 36 equally likely outcomes in the sample space, as shown in Figure 2.

Figure 2



So, $N(S) = 36$. The event E = "roll a seven" = $\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ has six outcomes, so $N(E) = 6$. Using Formula (3),

$$P(E) = P(\text{roll a seven}) = \frac{N(E)}{N(S)} = \frac{6}{36} = \frac{1}{6}$$

(b) The event F = "roll a two" = $\{(1,1)\}$ has one outcome, so $N(F) = 1$.

$$P(F) = P(\text{roll a two}) = \frac{N(F)}{N(S)} = \frac{1}{36}$$

(c) Since $P(\text{roll a seven}) = \frac{6}{36}$ and $P(\text{roll a two}) = \frac{1}{36}$, rolling a seven is six times as likely as rolling a two. In other words, in 36 rolls of the dice, we *expect* to observe about 6 sevens and only 1 two.

Roll a pair of fair dice Continuous
Find the probability of the following events:

a. A: "Sum = 7"
b. B: "Sum = 11"
c. C: "Sum = 12"
d. D: "Sum < 5"

Sum	Prob.
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Soln: P(A) = 6/36, P(B) = 2/36, P(C) = 1/36, P(D) = 6/36 .

Toss two coins. List the elements of the sample space and find P("At least one head").

S={HH, HT, TH, TT} each having a probability of ¼
"At least one head" = { HH, HT, TH}
P("At least one head")=3/4 or 0.75 or 75%.

Toss three coins. List the elements of the sample space and find the probability of the following events:
S={HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

a. A: "(exactly) two heads" ={HHT, HTH, THH}
b. B: "at least two heads"= {HHH, HHT, HTH, THH}
c. C: "at most two heads" ={HHT, HTH, THH, TTH, THT, HTT, TTT}

Soln: P(A) = 3/8, P(B) = 4/8, P(C) = 7/8

Empirical or Relative Frequency Approach: The relative frequency approach of assigning probabilities is appropriate when the data are available to estimate the proportion of the time the "outcome" will occur when the experiment is repeated a large number of times. Note that this approach does not require that each experimental outcome is equally likely.

Method for approximating P(E): Conduct (or observe) an experiment a large number of times and count the # of times the event E actually took place.

$$P(E) \approx \frac{\#\ of\ \ times\ E\ occurred}{\#\ of\ \ times\ experiment\ was\ repeated}$$

<u>Note:</u> As the experiment is repeated again and again, the empirical probability of success tend to approach the actual probability.

<u>Example 5.</u> The final exam in a course resulted in the following grades

Grade	A	B	C	D	F
Number	7	12	16	5	3

a. What is the probability that a randomly selected student received an A? **7/43**

b. Verify that it's a probability model. $\frac{7}{43}+\frac{12}{43}+\frac{16}{43}+\frac{5}{43}+\frac{3}{43}=\frac{43}{43}=1$

<u>Definition:</u> An **<u>unusual event</u>** is an event that has low probability of occurring, less than 5% according to the book..

Homework-Section 5.1 Online - MyStatLab

Sections 5.2 Addition Rule and Complements

Union

The union of two events A and B is the event containing all sample points in A or B or both. **Notation: $A \cup B$**. Also, the words **OR** and **At Least one** are associated with the union.

Intersection

The intersection of two events A and B is the event composed of all sample points that are in both A and B. **Notation: $A \cap B$.** Also, the words **and** and **Both** are associated with the intersection.

<u>**Note:**</u> $P(A \cup B)$ = P(A or B) = P(event A occurs or event B occurs or they both occur)

$P(A \cap B)$ = P(A and B) = P(event A and B both occur)

Complement of an Event: If A is an event over the sample space S, the complement of A **(Notation: A^C)** is defined to be the event consisting of all sample points in S that are not in A.

Subtraction Rule: $P(A) + P(A^C) = 1$

That is, $P(A^C) = 1 - P(A)$ or $P(A) = 1 - P(A^C)$

Example: Experiment. Roll a die once. S = { 1, 2, 3, 4, 5, 6} .

We defined the following events: A = {2, 4, 6} . P(A) = 3/6

B = {1, 3, 5 }. P(B) = 3/6

C = {1, 2, 3, 4} . P(C) = 4/6

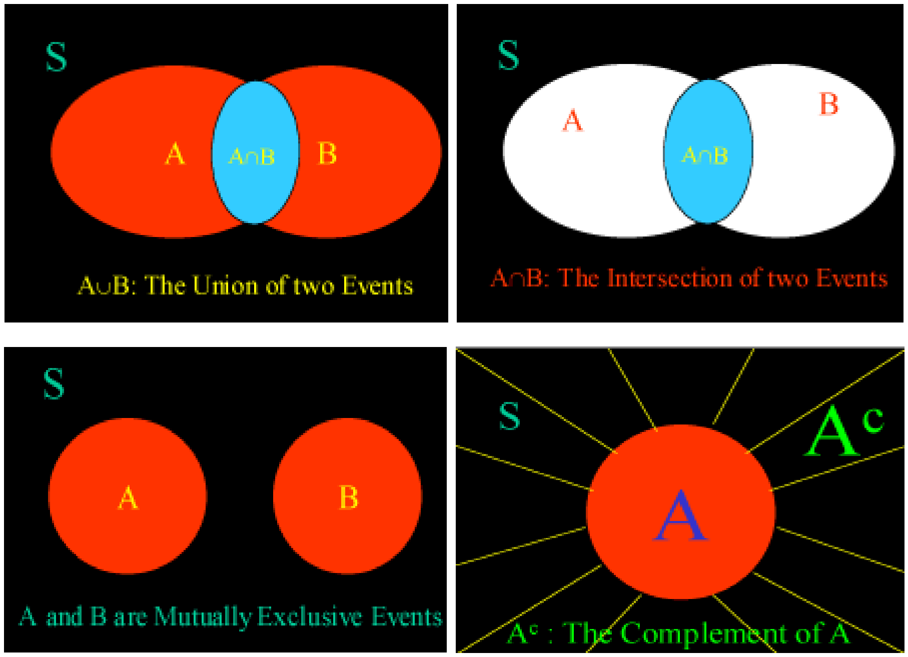
$A \cup B$ = { 1, 2, 3, 4, 5, 6} $P(A \cup B)$ = 1/6+ 1/6+1/6+1/6+1/6+1/6= 6/6 = 1.
$A \cup C$ = { 1, 2, 3, 4, 6 } $P(A \cup C)$ = 1/6+ 1/6+1/6+1/6+1/6 = 5/6.
$B \cup C$ = { 1, 2, 3, 4, 5 } $P(B \cup C)$ = 1/6+ 1/6+1/6+1/6+1/6 = 5/6.

$A \cap B = \varnothing$ (Null set} $P(A \cap B)$ = 0.
$A \cap C$ = { 2, 4 } $P(A \cap C)$ = 2/6.
$B \cap C$ = { 1, 3 } $P(B \cap C)$ = 2/6.

A= { 2, 4, 6 } A^c = { 1, 3, 5 } $P(A^c)$= 3/6 = 1 - P(A)= 1 - 3/6 = 3/6

NOTE: $A \cup A^c$ = S (sample space) and $A \cap A^c = \varnothing$.

Using Venn Diagrams: Union, Intersection, and Complement



Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ Please note that the addition rule is associated with the union.

Example: In a study of 100 students that had been awarded university scholarships, it was found that 40 had part-time jobs, 25 had made the dean's list the previous semester, and 15 had both a part-time job and had made the dean's list. What was the probability that a student had a part-time job or was on the dean's list? Soln: **50/100 or 0.5 or 50%**

Soln: $P(A \cup B)=P(A)+P(B)-P(A \cap B) = (40/100)+(25/100)-(15/100) = 50/100 = 0.5$

Example: You are playing a card game in which spades and honors (Ace, Queen, King, or Jack) are valuable. If you draw a card from the full deck, what is the probability that it is a "valuable card?" Soln: **25/52**

Soln: A "valuable card" is either a spade or Honors, there for we have a union. **$P(S \cup H)=P(S)+P(H)-P(S \cap H) = (13/52)+(16/52)-(4/52) = 25/52$**

Mutually Exclusive Events: ("ME" or "Disjoint sets") Two events A and B are called mutually exclusive if $A \cap B$ contains no sample points. That is, A and B have no outcomes in common. The probability that both will occur is Zero. It's either A occurs or B occurs but NOT both.

Note: If A and B are mutually exclusive, then $P(A \cap B) = 0$.

Addition Rule for "ME" or "Disjoint" Events
If A and B are "ME" or "Disjoint", then $P(A \cup B) = P(A) + P(B)$

Example: If one card is drawn from a full deck of 52 cards, what is the probability that the card drawn is a Queen or a Six? **Soln. 2/13**

Since find a card that is either a queen or a six are mutually exclusive or disjoint **$P(Q \cup 6)=P(Q)+P(6) = (4/52)+(4/52) - 0 = 8/52 = 2/13$**

The Table below is called a **contingency table** or **two-way table**, because it relates two categories of data. The row variable is marital status, because each row in the table describes the marital status of an individual. The column variable is gender. Each box inside the table is called a cell and it's an intersection. For example, the cell corresponding to married individuals who are male is in the second row, first column. Each cell contains the frequency of the category: There were 62.2 million married males in the United States in 2013. Put another way, in the United States in 2013, there were 62.2 million individuals who were male and married.

		Gender		
		Male	Female	Totals
Marital Status	Never Married	40.2	34.0	74.2
	Married	62.2	62.0	124.2
	Widowed	3.0	11.4	14.4
	Divorced	10.0	13.4	23.4
	Separated	2.4	3.2	5.6
	Totals	117.8	124.0	241.8

Question: How many individuals are female? **124.0 million**
Question: How many individuals are Divorced? **23.4 million**
Question: How many individuals are female and divorced? **13.4 million**
Question: How many individuals are married and divorced? **Zero, since they are mutually exclusive**

We can divide every number in the table above by the total of 241.8 million and turn it into a probability table. (see table below)

		Gender		
		Male	Female	Totals
Marital Status	Never Married	0.166	0.141	0.307
	Married	0.257	0.256	0.513
	Widowed	0.013	0.047	0.060
	Divorced	0.041	0.056	0.097
	Separated	0.010	0.013	0.023
	Totals	0.487	0.513	1

Now we can answer probability questions.

Question: What is the probability, that a randomly select individual is female? **0.513 directly from the table, or 124/241.8=0.513**

Question: What is the probability, that a randomly select individual is divorced? **0.097 directly from the table**

Question: What is the probability, that a randomly select individual is a female and divorced? **0.056 directly from the table**

Question: What is the probability, that a randomly select individual is married **and** divorced? **0 or 0%**

Question: What is the probability, that a randomly select individual is **F**emale **or D**ivorced?

$$\mathbf{P(F \cup D)=P(F)+P(D)-P(F \cap D) = 0.513 + 0.097 - 0.056 = 0.554}$$

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Section 5.3 Independence and the Multiplication Rule

Independent Events: Two events A and B are **independent** if the occurrence of event A in a probability experiment **does not affect** the probability of event B.

Dependent Events: Two events A and B are **dependent** if the occurrence of event A in a probability experiment **affects** the probability of event B.

Multiplication Rule for independent events
P(A and B) = P(A)P(B) also $P(A \cap B) = P(A)P(B)$

Example: Flip a coin twice. The probability of finding two heads is 0.25 since the sample space is S = {HH, HT, TH, TT}; i.e. P(HH)= 0.25.

Now, we can use the multiplication rule for independent events to show that the P(HH)=0.25. $P(H \cap H) = P(H)P(H)=(0.5)(0.5)=0.25$.

Life Expectancy

Multiplication Rule for *n* Independent Events

If events $E_1, E_2, E_3, \ldots, E_n$ are independent, then

$$P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots \text{ and } E_n) = P(E_1) \cdot P(E_2) \cdot \cdots \cdot P(E_n)$$

Life Expectancy

Problem The probability that a randomly selected 24-year-old male will survive the year is 0.9986 according to the *National Vital Statistics Report*, Vol. 56, No. 9. (a) What is the probability that three randomly selected 24-year-old males will survive the year? (b) What is the probability that 20 randomly selected 24-year-old males will survive the year?

Approach It is safe to assume that the outcomes of the probability experiment are independent, because there is no indication that the survival of one male affects the survival of the others.

Solution

(a) $P(\text{all three males survive}) = P(\text{1st survives and 2nd survives and 3rd survives}$

$= P(\text{1st survives}) \cdot P(\text{2nd survives}) \cdot P(\text{3rd survive}$

$= (0.9986)(0.9986)(0.9986)$

$= 0.9958$

If we randomly selected three 24-year-old males 1000 different times, we would expect all three to survive one year in 996 of the samples.

(b) $P(\text{all 20 males survive}) = P(\text{1st survives and 2nd survives and ... and 20th surviv}$

$= P(\text{1st survives}) \cdot P(\text{2nd survives}) \cdot \cdots \cdot P(\text{20th survive}$

$= \underbrace{(0.9986) \cdot (0.9986) \cdot \cdots \cdot (0.9986)}_{\text{Multiply 0.9986 by itself 20 times.}}$

$= (0.9986)^{20}$

$= 0.9724$

If we randomly selected twenty 24-year-old males 1000 different times, we woul expect all twenty to survive one year in 972 of the samples.

Computing At-Least Probabilities

Problem Compute the probability that at least one male out of 1000 aged 24 years will die during the course of the year if the probability that a randomly selected 24-year-old male survives the year is 0.9986.

Approach The phrase *at least* means "greater than or equal to," so we wish to know the probability that 1 or 2 or 3 or . . . or 1000 males will die during the year. These events are mutually exclusive, so

$$P(\text{1 or 2 or 3 or ... or 1000 die}) = P(\text{1 dies}) + P(\text{2 die}) + P(\text{3 die}) + \cdots + P(\text{1000 die})$$

Computing these probabilities would be very time consuming. However, notice that the complement of "at least one dying" is "none die", or all 1000 survive. Use the Complement Rule to compute the probability.

Solution

$P(\text{at least one dies}) = 1 - P(\text{none die})$

$= 1 - P(\text{1st survives and 2nd survives and ... and 1000th survives})$

$= 1 - P(\text{1st survives}) \cdot P(\text{2nd survives}) \cdot \cdots \cdot P(\text{1000th survives})$ Independent events

$= 1 - (0.9986)^{1000}$

$= 1 - 0.2464$

$= 0.7536$

If we randomly selected 1000 males 24 years of age 100 different times, we would expect at least one to die in 75 of the samples.

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Section 5.4 Conditional Probability:

$$P\langle A|B\rangle = \frac{P(A\cap B)}{P(B)} \qquad \textbf{or} \qquad P\langle B|A\rangle = \frac{P(A\cap B)}{P(A)}$$

Birth Weights of Preterm Babies

Problem Suppose that 12.7% of all births are preterm. (The gestation period of the pregnancy is less than 37 weeks.) Also 0.22% of all births resulted in a preterm baby who weighed 8 pounds, 13 ounces or more. What is the probability that a randomly selected baby weighs 8 pounds, 13 ounces or more, given that the baby is preterm? Is this unusual? *Source:* Vital Statistics Reports

Approach We want to know the probability that the baby weighs 8 pounds, 13 ounces or more, given that the baby was preterm. Because 0.22% of all babies weigh 8 pounds, 13 ounces or more and are preterm, $P(\text{weighs 8 lb, 13 oz or more and preterm}) = 0.0022$. Since 12.7% of all births are preterm, $P(\text{preterm}) = 0.127$. The phrase "given that" suggests we use the Conditional Probability Rule to compute the probability.

Solution $P(\text{weighs 8 lb, 13 oz or more}|\text{preterm})$

$$= \frac{P(\text{weighs 8 lb, 13 oz or more and preterm})}{P(\text{preterm})}$$

$$= \frac{0.0022}{0.127} \approx 0.0173$$

If 100 preterm babies were randomly selected, we would expect about two to weigh 8 pounds, 13 ounces or more. This is an unusual outcome. •

Example: A major metropolitan police force in the eastern United States consists of 1200 officers, 960 men and 240 women. Over the past 2 years, 324 officers on the police force have been awarded promotions. The breakdown of the promotions is given in the following table. A committee of female officers claimed discrimination. Are they right or wrong?

		Gender		
		Male	**Female**	Totals
Promotion	**Promoted(P)**	288	36	324
	Not Promoted(P^c)	672	204	876
	Totals	960	240	1200

First divide everything by the total of 1200 to turn it into a probability table (see below).

		Gender		Totals
		Male	Female	
Promotion	Promoted	.24	.03	.27
	Not Promoted	.56	.17	.73
	Totals	.80	.20	1

The two questions of proving or disproving discrimination is given below.

$$P(P \mid M) = \frac{P(M \cap P)}{P(M)} = \frac{0.24}{0.8} = 0.30$$

$$P(P \mid F) = \frac{P(F \cap P)}{P(F)} = \frac{0.03}{0.2} = 0.15$$

Clearly, given that you are a male you have twice as much chance of getting promoted. Hence, the ladies were right to claim discrimination.

General Multiplication Rule (When the events are **Dependent**)

$P(A \cap B) = P(A)P(B|A)$ or $P(A \cap B) = P(B)P(A|B)$

Multiplication Rule (When the events are **Independent**)

$P(A \cap B) = P(A)P(B)$

Example: A service station manager knows from past experience that 80% of the customers use a credit card when purchasing gasoline. What is the probability that the next two customers will both pay with a credit card?

Independent Events

$P(C1 \cap C2) = P(C1)\ P(C2) = (0.8)(0.8) = 0.64$ or 64%

Example: I would like to purchase my dream house. My chance of getting promoted is 60%. The probability of buying the house if I get promoted is 80%. What is the probability of getting promoted and buying the house?

Dependent Events

$P(P \cap BH) = P(P)\ P(BH|P) = (0.60)(0.80) = 0.48$ or 48%

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