CHAPTER 6

Section 6.1 Estimating with confidence

From the **C.L.T.** we know that if $n \geq 30$, then $\bar{X}$ is approximately Normally distributed with mean, μ and S.D. $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$. That is, $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$. If σ is unknown use the sample standard deviation **S**; i.e. $\bar{X} \sim N\left(\mu, \frac{S}{\sqrt{n}}\right)$.

A parameter of a population is an unknown constant, for example: μ. We are going to use the sample statistic, $\bar{X}$, to estimate the unknown population parameter μ.

Point estimation: It is the procedure involving the computation of a sample statistic in order to estimate a population parameter.

$\bar{X}$ is a sample statistic, used to estimate μ. $\bar{X}$ is the **point estimator** of the population parameter μ ; The actual numerical value obtained for $\bar{X}$ is called the **point estimate** of μ.

Properties of point estimators: One of the desired properties is for the estimator to be unbiased. Whenever the expected value of the estimator is equal to the value of the corresponding population parameter, the estimator is said to be an unbiased estimator. We showed that $E(\bar{X}) = \mu$. That is, $\bar{X}$ is an unbiased estimator of μ.

Sampling Error: We would like to know how good the estimator is. $Sampling\ Error = |\bar{X} - \mu|$. Note, this is impossible unless μ is known. Since $\bar{X}$ is approximately Normally distributed with mean μ and standard deviation $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$, we can use this fact to make probabilistic statements about the sampling error.

Probability statement about the sampling error: There is a $(1-\alpha)$ probability that the value of a sample mean will provide a sampling error of $Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ or less.

Interval Estimation of a Population Mean.(Large Sample Case, $n \geq 30$)
We will use the point estimate with the probability information about the sampling error to obtain an interval estimate of the population mean μ. This interval is called $(1-\alpha)\%$ **Confidence Interval** and is given by $\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ if σ is known or $\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$ if σ is unknown .

NOTE: $(1-\alpha)\%$ is called the confidence level. We are $(1-\alpha)\%$ confident that the interval will capture the true population mean μ and $(\alpha)\%$ that it will fail.

Do example 6.4, 6.5, 6.6 pp. 350-352

EXAMPLE 6.4 Average college savings fund contribution. One survey question asked how much money from a college savings fund, such as a 529 plan, is used to pay for college. In a sample of n = 1593 the average amount is $1768.
Let's compute an approximate 95% confidence interval for the true mean amount contributed from a college savings fund among all undergraduates. We'll assume that the standard deviation for the population of college savings fund contributions is $1483. For 95% confidence, we see from Table D that z^* = 1.960. The 95% confidence interval for μ is,

$$\bar{X} \pm m \Rightarrow \bar{X} \pm Z_{\frac{\alpha}{2}} * \frac{S}{\sqrt{n}} \Rightarrow 1768 \pm 1.96 * \frac{1483}{\sqrt{1593}} \Rightarrow 1768 \pm 1.96 * \frac{1483}{\sqrt{1593}}$$

$$\Rightarrow 1768 \pm 72.83 \Rightarrow (1768 - 72.83,\ \ 1768 + 72.83) \Rightarrow (1695.17,\ \ 1840.83)$$

We are 95% confident that the mean amount contributed from a college savings fund among all undergraduates is between $1695.17 and $1840.83.

EXAMPLE 6.5 How sample size affects the confidence interval. As in Example 6.4, the sample mean of the college savings fund contribution is $1768 and the population standard deviation is $1483. Suppose that the sample size is only 177 but still large enough for us to rely on the central limit theorem. In this case, the 95% confidence interval for μ is,

$$\bar{X} \pm m \Rightarrow \bar{X} \pm Z_{\frac{\alpha}{2}} * \frac{S}{\sqrt{n}} \Rightarrow 1768 \pm 1.96 * \frac{1483}{\sqrt{177}} \Rightarrow 1768 \pm 1.96 * \frac{1483}{\sqrt{177}}$$

$$\Rightarrow 1768 \pm 218.48 \Rightarrow (1768 - 218.48,\ \ 1768 + 218.48) \Rightarrow (1549.52,\ \ 1986.48)$$

We are 95% confident that the mean amount contributed from a college savings fund among all undergraduates is between $1549.52 and $1986.48.

EXAMPLE 6.6 How the confidence level affects the confidence interval. Suppose that for the college saving fund contribution data in Example 6.4 , we wanted 99% confidence. Table D tells us that for 99% confidence, z^* = 2.576. The 99% confidence interval for μ is,

$$\bar{X} \pm m \Rightarrow \bar{X} \pm Z_{\frac{\alpha}{2}} * \frac{S}{\sqrt{n}} \Rightarrow 1768 \pm 2.575 * \frac{1483}{\sqrt{1593}} \Rightarrow 1768 \pm 2.575 * \frac{1483}{\sqrt{1593}}$$

$$\Rightarrow 1768 \pm 95.68 \Rightarrow (1768 - 95.68,\ \ 1768 + 95.68) \Rightarrow (1672.32,\ \ 1863.68)$$

Requiring 99%, rather than 95%, confidence has increased the margin of error from 72.83 to 95.68.

Example 3: A sample of 36 parts was assembled using a proposed production method. Suppose that the sample mean time to assemble a part is $\bar{X}$ **=15.3** minutes, and the sample standard deviation is **S=1.3** minutes. The objective is to develop a 98% C.I. estimate for the population mean Time to assemble a part. **Note**: $Z_{\frac{\alpha}{2}} = Z_{\frac{.02}{2}} = Z_{.01} = 2.33$

$$\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \Rightarrow 15.3 \pm (2.33)(\frac{1.3}{\sqrt{36}}) \Rightarrow 15.3 \pm 0.5 \Rightarrow (14.8, 15.8)$$

Thus we are 98% confident that the true population mean μ is between **14.8 to 15.8** minutes.

How can we make the C.I. narrower?
(a) By increasing the sample size and (b) Decreasing the confidence level.

Determining the size of the sample

We know that the maximum sampling error; **m** = $Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$.

Solving the above equation for the sample size n, we get $n = \left(\frac{Z_{\frac{\alpha}{2}} \sigma}{m} \right)^2$.

EXAMPLE 6.7 p. 354 How many undergraduates should we survey? Suppose that we are planning a survey. If we want the margin of error for the average amount contributed from a college savings plan to be \$30 with 95% confidence, what sample size *n* do we need? For 95% confidence, Table D gives $z^* = 1.960$. For σ we will use the value from the previous study, \$1483. If the margin of error is \$30, we have

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \sigma}{m} \right)^2 = \left(\frac{1.96 * 1483}{30} \right)^2 = 9387.542914 \approx 9388$$

Please Note: We ALWAYS round-up regardless of the decimal.

Would we need a much larger sample size to obtain a margin of error of \$25? Here is the calculation:

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \sigma}{m} \right)^2 = \left(\frac{1.96 * 1483}{25} \right)^2 = 13518.0618 \approx 13519.$$

Yes, we are going to need a sample of n = 13,519 in order to obtain a margin of error of \$25.

HW: 6.12, 6.13, 6.14, 6.17, 6.22, 6.24, 6.27, 6.28, 6.33, 6.34, 6.35, 6.36 pp. 357-360.

Section 6.2 HYPOTHESIS TESTS ABOUT A POPULATION MEAN

STATISTICAL INFERENCE: It's the process of drawing conclusions about a population parameter based on information contained in a sample. **NOTE:** Point Estimation, Interval Estimation and Hypothesis Testing go under statistical inference.

HYPOTHESIS TESTING

To do hypothesis testing the random variable $\bar{X}$ has to have a Normal distribution or approximately Normal and the sample selected to be a SRS.

1. First we make an assumption about the population mean(μ). This assumption is called **Null Hypothesis** and is denoted by H_0.
2. Then we define another Hypothesis, called the **Alternative Hypothesis** which is the opposite of what is stated in the Null Hypothesis. It's denoted by H_a .

Examples on How to set up the Hypothesis	
1. Ho: The defendant is innocent Ha: The defendant is guilty.	2. Ho: The mean life of a tire is ≥ 50,000 miles Ha: The mean life of a tire is < 50,000 miles.

In Hypothesis Testing we have three different cases.		
$Case 1.$ $H_0 : \mu \geq \mu_0$ $H_a : \mu < \mu_0$	$Case 2.$ $H_0 : \mu \leq \mu_0$ $H_a : \mu > \mu_0$	$Case 3.$ $H_0 : \mu = \mu_0$ $H_a : \mu \neq \mu_0$
where μ_0 is the Hypothesized value.		

NOTE: Cases 1 & 2 are called one tail tests. Case 1 is called the Lower tail test and Case 2 is called the Upper tail test. Case 3 is called a two tail test.

Some books present the three different cases of Hypothesis Testing as follow:

In Hypothesis Testing we have three different cases.		
$Case 1.$ $H_0 : \mu = \mu_0$ $H_a : \mu < \mu_0$	$Case 2.$ $H_0 : \mu = \mu_0$ $H_a : \mu > \mu_0$	$Case 3.$ $H_0 : \mu = \mu_0$ $H_a : \mu \neq \mu_0$
Where μ_0 is the Hypothesized value.		

Hypothesis Testing can be done in four Steps (In general).

Step 1. Null Hypothesis (H_0)
Alternative Hypothesis (H_a)
Step 2. Test statistic
Step 3. Critical region or Rejection region.
Step 4. Conclusion

General form of Hypothesis Testing

Step 1	$Case 1.\quad H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$Case 2.\quad H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$Case 3.\quad H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$
Step 2	$Z^* = \dfrac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$	$Z^* = \dfrac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$	$Z^* = \dfrac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$
Step 3	$\text{Re}\,ject\ H_0\ if\ Z^* < -Z_\alpha$	$\text{Re}\,ject\ H_0\ if\ Z^* > Z_\alpha$	$\text{Re}\,ject\ H_0\ if\ \lvert Z^* \rvert > Z_{\frac{\alpha}{2}}$
Step 4	Conclusion: We do not reject H_0 and conclude that .... We reject H_0 and conclude that		

NOTE: The **P-Value** is another way of drawing a conclusion. The **P-Value** is the probability of obtaining the test statistic under the assumption that the Null Hypothesis is true (H_0 is true).

$$If\ the\ P-Value \geq \alpha,\ then\ we\ failed\ to\ reject\ H_0\ .$$

$$If\ the\ P-Value < \alpha,\ then\ we\ reject\ H_0\ .$$

That is, if the probability of obtaining the test statistic is greater than or equal to α, $\overline{X}$ is not one of the extreme values but one that is close to μ.

$$Compute\ the\ P-Value: Let\ Z^* = Test\ Statistic;\quad That\ is\ Z^* = Z = \frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

$$Case 1:\quad P-Value = P\left(Z < Z^*\right)$$

$$Case 2:\quad P-Value = P\left(Z > Z^*\right)$$

$$Case 3:\quad P-Value = 2P\left(Z < -\left|Z^*\right|\right)$$

Example 1. A tire corp. produces tires with standard deviation σ=10,000 miles. A sample of size n = 100 tires was selected with a sample mean $\bar{X}$ = 48,100 miles. The warranty on the tires is 50,000 miles. The customers complaint that the mean life on the tires is less than 50,000 miles. Do the appropriate Hypothesis Testing at α=.05 level of significance.

1. $H_0 : \mu = 50,000 \quad vs \quad H_a : \mu < 50,000 \quad (Lower\ Tail\ Test)$
2. Test Statistic: $Z^* = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{48100 - 50000}{\frac{10000}{\sqrt{100}}} = -1.90$
3. $\text{Re}\,ject\ H_0\ if\ Z^* < -Z_{\alpha=0.05} = -1.645$
4. **Conclusion:** Since Z^* = -1.9 is less than -1.645, we reject H_0 and conclude that the mean life of a tire is less than 50,000 miles.
Note: P-Value = P(Z < Z*) = P(Z < -1.90) = .0287 P-Value = .0287 < α = 0.05. Therefore, we reject H_0 and draw the same conclusion as in step 4 above.

NOTE: The level of significance is the probability of rejecting the Null Hypothesis when in fact is true.

Example 2: Using example 1, we decided to test the hypothesis that the mean life of a tire is different than 50,000 miles.

1. $H_0 : \mu = 50,000 \quad vs \quad H_a : \mu \neq 50,000 \quad (Two\ Tail\ Test)$
2. Test Statistic: $Z^* = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{48100 - 50000}{\frac{10000}{\sqrt{100}}} = -1.90$
3. $\text{Re}\,ject\ H_0\ if\ \lvert Z^* \rvert > Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$
4. **Conclusion:** Since $\lvert Z^* \rvert = 1.9$ is less than 1.96, we fail to reject H_0 and conclude that the mean life of a tire is 50,000 miles.
Note: P-Value = $2P\left(Z < -\lvert Z^* \rvert\right)$ = 2P(Z < -1.90) = 2(.0287)=0.0574 P-Value = 0.0574 > α = 0.05. Therefore, we fail to reject H_0 and draw the same conclusion as in step 4 above.

<u>Example 3:</u> A school administrator has developed an individualized reading-comprehension program for eighth grade students. To evaluate this new program, a random sample of 45 eighth grade students was selected; these students participated in the new reading program for one semester and then took a standard reading-comprehension examination. The mean test score for the population of students who had taken this test in the past was 76. The S.R.S. of 45 students produced $\bar{X}$=79 and s=8.

Note: If the exercise does not say anything about the α -level of significance, we take it to be α = .05.

1. $H_0 : \mu = 76 \quad vs \quad H_a : \mu > 76 \ (Upper\ Tail\ Test)$
2. Test Statistic: $Z^* = \dfrac{\bar{X} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} = \dfrac{79 - 76}{\dfrac{8}{\sqrt{45}}} = 2.52$
3. $\text{Re}\,ject\ H_0\ if\ Z^* > Z_{\alpha=0.05} = 1.645$
4. **<u>Conclusion:</u>** Since Z^*= 2.52 is greater than 1.645, we reject H_0 and conclude that the new reading-comprehension program has a higher mean and should be implemented.
Note: P-Value = P(Z > Z*) = P(Z >2.52)=1- P(Z < 2.52)=1-0.9941 = 0.0059 P-Value = 0.0059 < α = 0.05. Therefore, we reject H_0 and draw the same conclusion as in step 4 above.

<u>Homework:</u> 6.58, 6.59, 6.72, 6.73, 6.74, 6.75 pp. 379-383