Section 1.3 The Probability Set Function

Let $\mathscr{C}$ be the sample Space, and $C \subset \mathscr{C}$, then $P(C) =$ The probability that the out of the random experiment is an element of C.

We saw the $P(C)$ to be the number to where the **f / N** of the event C tends to stabilize.

Definition: Let $C \subset \mathscr{C}$**, if**

(a) $P(C) \geq 0$

(b) $P(C_1 \cup C_2 \cup C_3 \cup \ldots) = P(C_1) + P(C_2) + P(C_3) + \ldots$

Where $P(C_i \cap C_j) = \varnothing$, $and\ \ i \neq j$

(c) $P(\mathscr{C}) = 1$

Then the $P(C)$ is called the probability set function.

Theorem 1: For each $C \subset \mathscr{C},\ P(C) = 1 - P(C^c)$.

Proof: $\mathscr{C} = C \cup C^c$ and $C \cap C^c = \varnothing$

$P(\mathscr{C}) = P(C) + P(C^c) \Rightarrow P(C) = P(\mathscr{C}) - P(C^c)$

$\Rightarrow P(C) = 1 - P(C^c)$ since $P(\mathscr{C}) = 1$.

Theorem 2: The probability of the null set is zero. That is, $P(\varnothing) = 0$.

Proof: $\mathscr{C} = C \cup C^c$ and $C \cap C^c = \varnothing$

Let $C = \varnothing$, then $C^c = \mathscr{C}$

$\mathscr{C} = C \cup C^c \Rightarrow$

$P(\mathscr{C}) = P(C) + P(C^c) \Rightarrow 1 = P(C) + P(C^c)$

$\Rightarrow P(C) = 1 - 1 = 0 \Rightarrow P(\varnothing) = 0$ since $C = \varnothing$

Theorem 3: If C_1 and C_2 are subsets of $\mathcal{C}$ such that $C_1 \subset C_2$ then $P(C_1) \le P(C_2)$.

Proof: Let $C_2 = C_1 \cup (C_1^c \cap C_2)$ then $C_1 \cap (C_1^c \cap C_2) = \varnothing$.

$P(C_2) = P(C_1) + P(C_1^c \cap C_2)$

If $P(C_1^c \cap C_2) = 0$ then $P(C_1) = P(C_2)$

If $P(C_1^c \cap C_2) > 0$ then $P(C_1) < P(C_2)$

Hence, $P(C_1) \le P(C_2)$

Theorem 4: For each $C \subset \mathcal{C}$, then $0 \le P(C) \le 1$.

Proof: Since $\varnothing \subset C \subset \mathcal{C}$ by theorem 3 $P(\varnothing) \le P(C) \le P(\mathcal{C}) \Rightarrow 0 \le P(C) \le 1$

Theorem 5: If C_1 and C_2 are subsets of $\mathcal{C}$ then $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$.

Proof: Let $C_1 \cup C_2 = C_1 \cup (C_1^c \cap C_2)$ the union of two non-intersecting sets an

$P(C_1 \cup C_2) = P(C_1) + P(C_1^c \cap C_2)$; Note: $P(C_1^c \cap C_2) = P(C_2) - P(C_1 \cap C_2)$

Hence, $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$

If the sets $C_1, C_2, C_3, \ldots$ are subsets of $\mathcal{C}$ such that no two sets have an element in common, they are called mutually exclusive sets. Furthermore, if $\mathcal{C} = C_1 \cup C_2 \cup C_3 \cup \ldots$, then they are also called mutually exclusive and exhaustive sets.

Example: Roll a die once. $\mathscr{C} = \{1,2,3,4,5,6\}$

$C_1 = \{1, 3, 5\}$ *and* $C_2 = \{2, 4, 6\}$ *and* $\mathscr{C} = C_1 \cup C_2$. Therefore, C_1 *and* C_2 are mutually exclusive and exhaustive sets.

Example HW 3.3 page 18. $\mathscr{C} = \{c : \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots\}$

a. $P(\mathscr{C}) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \sum_{i=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^i$. This is the geometric series with $a = \frac{1}{2}$ *and* $r = \frac{1}{2}$. $\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$.

b. Let $C_1 = \{H, TH, TTH, TTTH, TTTTH\}$; $P(C_1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

c. $C_2 = \{TTTTH, TTTTTH\}$; $P(C_2) = \frac{1}{32} + \frac{1}{64} = \frac{3}{64}$;

$C_1 \cap C_2 = \{TTTTH\}$; $P(C_1 \cap C_2) = \frac{1}{32}$ and

$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2) = \frac{31}{32} + \frac{3}{64} - \frac{1}{32} = \frac{63}{64}$

Example HW 3.5 page 18. $\mathscr{C} = \{c : 0 < c < \infty\}$ and $C \subset \mathscr{C}$ evaluate the $P(C) = \int_C e^{-x}\,dx$. $P(\mathscr{C}) = \int_0^{\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{\infty} = -\lim_{x\to\infty} e^{-x} - \left(-e^{-0}\right) = 0 - (-1) = 1$

a. If $C = \{c : 4 < c < \infty\}$ then $P(C) = \int_4^{\infty} e^{-x}\,dx = -e^{-x}\Big|_4^{\infty} = -\lim_{x\to\infty} e^{-x} - \left(-e^{-4}\right) = 0 + e^{-4} = e^{-4}$

b. If $C^c = \{c : 0 < c < 4\}$ then $P(C) = \int_0^{4} e^{-x}\,dx = -e^{-x}\Big|_0^{4} = -e^{-4} - \left(-e^{-0}\right) = 1 - e^{-4}$

Or since $P(C^c) = 1 - P(C) = 1 - e^{-4}$.

c. $P(C \cup C^c) = P(C) + P(C) - P(C \cap C^c) = e^{-4} + \left(1 - e^{-4}\right) - 0 = 1$

Homework: 3.4, 3.6, 3.7, and 3.9(part a only) on p.p. 18

Section 1.3 Continuous

Let $\mathcal{C}$ be partitioned into **k** mutually exclusive and exhaustive events, $C_1, C_2, C_3, \ldots, C_k$ such that $C_1 \cup C_2 \cup C_3 \cup \ldots C_k = \mathcal{C}$. We assume that each event C_i ; $i = 1, \ldots, k$ has the same probability. $P(C_i) = \frac{1}{k}$; $i = 1, \ldots, k$. The events $C_1, C_2, C_3, \ldots, C_k$ are equally likely to occur.

Let the event **E** be the union of **r** of these mutually exclusive and exhaustive events, say $E = C_1 \cup C_2 \cup C_3 \cup \ldots C_r$; $r \le k$. Since they are mutually exclusive events, $P(E) = P(C_1) + P(C_2) + P(C_3) + \ldots + P(C_r) = \frac{r}{k}$.

Example: We have an ordinary deck of cards, (k =52). Select one card and let E_1 the outcome is a spade, (r =13). $P(E_1) = \frac{r}{k} = \frac{13}{52}$.
Let E_2 the outcome is a king, (r =4). $P(E_2) = \frac{r}{k} = \frac{4}{52}$.

Now select 5 cards one at a time without replacement. What is the probability that all 5 cards are spade?

$$r = \binom{13}{5}\binom{39}{0} \quad and \quad k = \binom{52}{5}; \quad Hence, \; P(E_3) = \frac{r}{k} = \frac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}} = 0.000495$$

What is the probability that at least one card is a spade?

$$r = \binom{13}{1}\binom{39}{4} + \binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0} \quad and \quad k = \binom{52}{5};$$

$$Hence, \; P(E_4) = \frac{r}{k} = \frac{\binom{13}{1}\binom{39}{4} + \binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}}{\binom{52}{5}} = 0.7785$$

An easier way: Since the event **"at least"** always has the complement the **"none",** we can compute the probability for the complement and then use $P(C) = 1 - P(C^c)$.

$$r = \binom{13}{0}\binom{39}{5} \; and \; k = \binom{52}{5}; \quad Hence, \; P(E_5^c) = \frac{r}{k} = \frac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}} = 0.2215 \Rightarrow P(E) = 1 - 0.2215 = 0.7785$$

Select 5 cards. What is the probability of exactly 3 kings and 2 queens?

$$r = \binom{4}{3}\binom{4}{2}\binom{44}{0} \quad and \quad k = \binom{52}{5}; \quad Hence, \quad P(E) = \frac{r}{k} = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{0}}{\binom{52}{5}} = \frac{24}{2598960} = 0.000009$$

Select 5 cards. What is the probability of exactly 2 kings, 2 queens, and 1 jack?

$$r = \binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{40}{0} \quad and \quad k = \binom{52}{5}; \quad Hence, \quad P(E) = \frac{r}{k} = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{40}{0}}{\binom{52}{5}} = \frac{144}{2598960} = 0.000055$$

Example: Consider a loaded die. Means the dots on the die appear proportional to the sum of the total dots on the die. $\mathcal{C} = \{1,2,3,4,5,6\}$ and $f(x) = \frac{x}{21}$; $x = 1,2,3,4,5,6$

Roll the die once. What is the probability the number is even? $C = \{2, 4, 6\}$

$$P(C) = \frac{2+4+6}{21} = \frac{12}{21} = 0.5714$$

Homework: 3.10, 3.11, 3.12, 3.13, 3.14, 3.15(part a only) on p.p. 19