Section 1.4 Conditional Probability and Independence

Consider the sample space $\mathscr{C}$ with two events, C_1 and C_2. $P(C_2|C_1)$ is the conditional probability of C_2 occurring given that C_1 already occurred.

Since the event you condition with becomes your new sample space and the only elements you are interested are the elements that are common to both, $C_1 \cap C_2$, if any, then $P(C_1|C_1)=1$ and $P(C_2|C_1)=P(C_1 \cap C_2|C_1)$.

Example: Roll a regular die. $\mathscr{C}=\{1,2,3,4,5,6\}$, $C_1=\{2,4,6\}$, $C_2=\{2\}$ and $C_1 \cap C_2=\{2\}$. Note: $P(C_1)=\frac{3}{6}=\frac{1}{2}$; and $P(C_1|C_1)=1$.
$P(C_2)=\frac{1}{6}$; and $P(C_2|C_1)=P(C_1 \cap C_2|C_1)=\frac{1}{3}$

<u>**Relative to the sample space**</u> $\mathscr{C}$ $P(C_2|C_1)=\frac{P(C_1\cap C_2|C_1)}{P(C_1|C_1)}=\frac{P(C_1\cap C_2)}{P(C_1)}=\frac{\frac{1}{6}}{\frac{3}{6}}=\frac{1}{3}$.

Moreover if $C_1>0$, then $P(C_2|C_1)$ is a probability set function if

1. $P(C_2|C_1)>0$
2. $P(C_2 \cup C_3 \cup C_4 \cup \ldots|C_1)=P(C_2|C_1)+P(C_3|C_1)+P(C_4|C_1)+\ldots$
   provided $C_2, C_3, C_4, \ldots$ are mutually exclusive sets.
3. $P(C_1|C_1)=1$.

Proof: Parts 1 and 3 are obvious. Part 2 is exercise 4.1 on page 27.
Part 2 proof:

$$P(C_2 \cup C_3 \cup C_4 \cup \ldots|C_1)=\frac{P([C_2\cup C_3\cup C_4\cup \ldots]\cap C_1)}{P(C_1)}=\frac{P([(C_2\cap C_1)\cup(C_3\cap C_1)\cup(C_4\cap C_1)\cup\ldots])}{P(C_1)}$$

$\sin ce$ $C_2, C_3, C_4, \ldots$ *are mutually exclusive sets*

$$=\frac{P(C_2\cap C_1)}{P(C_1)}+\frac{P(C_3\cap C_1)}{P(C_1)}+\frac{P(C4\cap C_1)}{P(C_1)}+\ldots=P(C_2|C_1)+P(C_3|C_1)+P(C_4|C_1)+\ldots$$

Example 4.1 on page 22: A hand of five cards is to be dealt at random without replacement from an ordinary deck of 52 playing cards. What is the conditional probability of an all spade hand, C_2, relative to the hypothesis that there are at least four spaces in the hand, C_1? Note: $C_1 \cap C_2 = C_2$; *i.e.* $C_2 \subset C_1$.

C_2 = all 5 cards are spade; $P(C_2) = \dfrac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$

C_1 = at least 4 spades; $P(C_1) = \dfrac{\binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$

$$P(C_2|C_1) = \frac{P(C_1 \cap C_2)}{P(C_1)} = \frac{P(C_2)}{P(C_1)} = \frac{\frac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}}{\frac{\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}} = \frac{\binom{13}{5}\binom{39}{0}}{\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}} = 0.0441$$

Example 4.2 on page 22: A ball contains eight chips. Three of the chips are red and the remaining five are blue. Two chips are to be drawn successively, at random and without replacement. We want to compute the probability that the first draw results in a red chip (C_1) and that the second draw results in a blue chip (C_2).

$$P(C_1) = \tfrac{3}{8};\ \textit{and}\ \ P(C_2|C_1) = \tfrac{5}{7};$$

$$\textit{Hence}\ \ P(C_1 \cap C_2) = P(C_1)P(C_2|C_1) = \tfrac{3}{8} \times \tfrac{5}{7} = \tfrac{15}{56}$$

Question: Can we use the combinations method to solve this problem?

r = 1 red chip and 1 blue = $\binom{3}{1}\binom{5}{1}$ and $k = \binom{8}{2}$;

$P(E) = \dfrac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}} = \dfrac{3\times 5}{\frac{8!}{2!\times 6!}} = \dfrac{15}{\frac{8\times 7\times 6!}{2\times 6!}} = \dfrac{15}{\frac{8\times 7}{2}} = \dfrac{15}{28}$; Did not work. $\frac{15}{28} \neq \frac{15}{56}$ Why?

Example 4.3 on page 22: From an ordinary deck of playing cards, cards are to be drawn successively, at random and without replacement. Compute the probability that the third spade appears on the sixth draw.

Let C_1 be the event of two spades in the first five draws **and** let C_2 be the event of a spade on the sixth draw. Compute $P(C_1 \cap C_2)$.

$$P(C_1) = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} = 0.2743 \quad and \quad P(C_2|C_1) = \tfrac{11}{47} = 0.2340$$

$$Hence \quad P(C_1 \cap C_2) = P(C_1)P(C_2|C_1) = 0.2743 \times 0.2340 = 0.0642$$

Example 4.4 on page 23: Four cards are to be dealt successively, at random and without replacement, from an ordinary deck of playing cards. Compute the probability of receiving a spade, a heart, a diamond, and a club, in that order.

$$P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1)P(C_2|C_1)P(C_3|C_1 \cap C_2)P(C_4|C_1 \cap C_2 \cap C_3)$$
$$= \tfrac{13}{52} \times \tfrac{13}{51} \times \tfrac{13}{50} \times \tfrac{13}{49} = 0.0044$$

Homework: 4.4, 4.5, 4.6, 4.8, 4.9 on page 28

Section 1.4 Continuous – Bayes' Theorem

Bayes' Theorem: $P(C_j|C) = \frac{P(C \cap C_j)}{P(C)} = \frac{P(C_j)P(C|C_j)}{\sum_{i=1}^{k} P(C_i)P(C|C_i)}$

Proof: Partition the sample space $\mathcal{C}$ into **k** mutually exclusive and exhaustive events, $C_1, C_2, C_3, \ldots, C_k$ such that $P(C_i) \geq 0$; $i = 1, \ldots, k$.

Let C be another event such that $P(C) > 0$. Thus, C occurs with one and only one of the events $C_1, C_2, C_3, \ldots, C_k$; that is $C = C \cap (C_1 \cup C_2 \cup C_3 \cup \ldots \cup C_k)$

$$C = (C \cap C_1) \cup (C \cap C_2) \cup (C \cap C_3) \cup \ldots \cup (C \cap C_k)$$

$$P(C) = P(C \cap C_1) + P(C \cap C_2) + P(C \cap C_3) + \ldots + P(C \cap C_k)$$

$$P(C) = P(C_1)P(C|C_1) + P(C_2)P(C|C_2) + P(C_3)P(C|C_3) + \ldots + P(C_k)P(C|C_k)$$

$$P(C) = \sum_{i=1}^{k} P(C_i)P(C|C_i)$$

Hence, $P(C_j|C) = \frac{P(C \cap C_j)}{P(C)} = \frac{P(C_j)P(C|C_j)}{\sum_{i=1}^{k} P(C_i)P(C|C_i)}$

Example 4.5 on page 23: Bowl1 contains three red and seven blue chips and bowl2 contains eight red and two blue chips. A die is cast and bowl1 C_1 is selected if 5 or 6 is observed; otherwise , bowl2 C_2 is selected. The selected bowl is handed to another person and one chip is taken at random. Say that this chip is red denoted by the event C . Given that this chip is red, event C, what is the probability the red chip was drawn from bowl1 C_1? ; i.e. $P(C_1|C)$

Note: $P(C_1) = \frac{2}{6}$ *and* $P(C_2) = \frac{4}{6}$; *Also,* $P(C|C_1) = \frac{3}{10}$ *and* $P(C|C_2) = \frac{8}{10}$

$$P(C_1|C) = \frac{P(C_1 \cap C)}{P(C)} = \frac{P(C_1 \cap C)}{P(C_1 \cap C) + P(C_2 \cap C)} = \frac{P(C_1)P(C|C_1)}{P(C_1)P(C|C_1) + P(C_2)P(C|C_2)} = \frac{\frac{2}{6} \times \frac{3}{10}}{\left(\frac{2}{6} \times \frac{3}{10}\right) + \left(\frac{4}{6} \times \frac{8}{10}\right)} = \frac{6}{38} = \frac{3}{19}$$

$$P(C_2|C) = \frac{P(C_2 \cap C)}{P(C)} = \frac{P(C_2 \cap C)}{P(C_1 \cap C) + P(C_2 \cap C)} = \frac{P(C_2)P(C|C_2)}{P(C_1)P(C|C_1) + P(C_2)P(C|C_2)} = \frac{\frac{4}{6} \times \frac{8}{10}}{\left(\frac{2}{6} \times \frac{3}{10}\right) + \left(\frac{4}{6} \times \frac{8}{10}\right)} = \frac{32}{38} = \frac{16}{19}$$

Example 4.6 on page 23: Three plants, C_1, C_2, C_3, produce respectively, 10%, 50%, and 40% of company's output. Although plant C_1 is a small plant, its manager believes in high quality and only 1% of its products are defective. The other two, C_2 and C_3, are worse and produce items that are 3% and 4% defective, respectively. One item is selected at random and observed to be defective, say event C. What is the probability that the defective item came from plant C_1? ; i.e. $P(C_1|C)$

Note:

$P(C_1) = 0.1$, $P(C_2) = 0.5$, $P(C_3) = 0.4$; $P(C|C_1) = 0.01$, $P(C|C_2) = 0.03$, $P(C|C_2) = 0.04$

$$P\left(C_1|C\right)=\frac{P(C_1\cap C)}{P(C_1\cap C)+P(C_2\cap C)+P(C_3\cap C)}=\frac{P(C_1)P(C|C_1)}{P(C_1)P(C|C_1)+P(C_2)P(C|C_2)+P(C_3)P(C|C_3)}$$

$$=\frac{0.1\times 0.01}{(0.1\times 0.01)+(0.5\times 0.03)+(0.4\times 0.04)}=\frac{\frac{10}{100}\times\frac{1}{100}}{\left(\frac{10}{100}\times\frac{1}{100}\right)+\left(\frac{50}{100}\times\frac{3}{100}\right)+\left(\frac{40}{100}\times\frac{4}{100}\right)}=\frac{10}{10+150+160}=\frac{10}{320}=\frac{1}{32}$$

Independent events: If C occurs and the probability of C_1 does not change; i.e. $P\left(C_1|C\right)=P\left(C_1\right)$ then C_1 and C are independent events.

$$P\left(C_1|C\right)=\frac{P(C_1\cap C)}{P(C)}\Rightarrow P\left(C_1\cap C\right)=P\left(C\right)P\left(C_1|C\right)$$

$$\text{since } C_1 \text{ and } C \text{ are independent the } P\left(C_1|C\right)=P\left(C_1\right)\Rightarrow P\left(C_1\cap C\right)=P\left(C\right)P\left(C_1\right)$$

Note:

a. C_1 and C_2 are independent if $P\left(C_1\right)=0$ or $P\left(C_2\right)=0$, implies $P\left(C_1\cap C_2\right)=0$

Proof: Since $\left(C_1\cap C_2\right)\subset C_1$ *and* $\left(C_1\cap C_2\right)\subset C_2$, then if

$$P\left(C_1\right)=0\Rightarrow P\left(C_1\cap C_2\right)=0 \quad \text{or if} \quad P\left(C_2\right)=0\Rightarrow P\left(C_1\cap C_2\right)=0.$$

b. If C_1 and C_2 are independent so are their pairs: C_1 *and* C_2^c, C_1^c *and* C_2, C_1^c *and* C_2^c.

Proof:

$$P\left(C_1\cap C_2^c\right)=P\left(C_1\right)-P\left(C_1\cap C_2\right)=P\left(C_1\right)-P\left(C_1\right)P\left(C_2\right)$$
$$=P\left(C_1\right)\left(1-P\left(C_2\right)\right)=P\left(C_1\right)P\left(C_2^c\right)$$

Proof:

$$P\left(C_1^c\cap C_2\right)=P\left(C_2\right)-P\left(C_1\cap C_2\right)=P\left(C_2\right)-P\left(C_1\right)P\left(C_2\right)$$
$$=P\left(C_2\right)\left(1-P\left(C_1\right)\right)=P\left(C_2\right)P\left(C_1^c\right)$$

Proof:

$$P\left(C_1^c\cap C_2^c\right)=P\left(\left(C_1\cup C_2\right)^c\right)=1-P\left(C_1\cup C_2\right)$$
$$=1-\left[P\left(C_1\right)+P\left(C_2\right)-P\left(C_1\cap C_2\right)\right]=1-P\left(C_1\right)-P\left(C_2\right)+P\left(C_1\cap C_2\right)$$
$$=P\left(C_1^c\right)-P\left(C_2\right)+P\left(C_1\right)P\left(C_2\right)=P\left(C_1^c\right)-P\left(C_2\right)\left[1-P\left(C_1\right)\right]$$
$$=P\left(C_1^c\right)-P\left(C_2\right)\left[P\left(C_1^c\right)\right]=P\left(C_1^c\right)\left(1-P\left(C_2\right)\right)=P\left(C_1^c\right)P\left(C_2^c\right)$$

Mutually Independent: Three events C_1, C_2, C_3 are mutually independent if and only if they are pairwise independent, $P(C_1 \cap C_2) = P(C_1)P(C_2)$, $P(C_1 \cap C_3) = P(C_1)P(C_3)$, $P(C_2 \cap C_3) = P(C_2)P(C_3)$ and $P(C_1 \cap C_2 \cap C_3) = P(C_1)P(C_2)P(C_3)$.

Example 4.9 on page 27:

Pairwise Independence does not imply mutual independence

We spin twice a fair spinner with the numbers 1, 2, 3, and 4. Let C_1 be the event the sum of the two numbers is 5 , C_2 the event the first number spun is 1, and C_3 the second number spun is 4.

$$\mathcal{C} = \left\{ \begin{array}{l} (x,y);\ (1,1),(1,2),(1,3),(1,4) \\ \qquad\quad (2,1),(2,2),(2,3),(2,4) \\ \qquad\quad (3,1),(3,2),(3,3),(3,4) \\ \qquad\quad (4,1),(4,2),(4,3),(4,4) \end{array} \right\} \quad and \quad P(\mathcal{C}) = \frac{16}{16} = 1$$

$$C_1 = \{(x,y);\ (2,3),(3,2),(1,4),(4,1)\} \ \ and \ \ P(C_1) = \frac{4}{16} = \frac{1}{4}$$

$$C_2 = \{(x,y);\ (1,1),(1,2),(1,3),(1,4)\} \ \ and \ \ P(C_2) = \frac{4}{16} = \frac{1}{4}$$

$$C_3 = \{(x,y);\ (1,4),(2,4),(3,4),(4,4)\} \ \ and \ \ P(C_3) = \frac{4}{16} = \frac{1}{4}$$

Pairwise:

$$C_1 \cap C_2 = \{(x,y);\ (1,4)\} \ \Rightarrow P(C_1 \cap C_2) = P(C_1) * P(C_2) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

$$C_1 \cap C_3 = \{(x,y);\ (1,4)\} \ \Rightarrow P(C_1 \cap C_3) = P(C_1) * P(C_3) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

$$C_2 \cap C_3 = \{(x,y);\ (1,4)\} \ \Rightarrow P(C_2 \cap C_3) = P(C_2) * P(C_3) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

Hence, C_1, C_2, and C_3 are pairwise independent but not mutually independent since

$$C_1 \cap C_2 \cap C_3 = \{(x, y);\ (1, 4)\}$$

$$\Rightarrow P(C_1 \cap C_2 \cap C_3) = \frac{1}{16} \quad \neq \quad \frac{1}{64} = P(C_1) * P(C_2) * P(C_3) = \frac{1}{4} * \frac{1}{4} * \frac{1}{4} = \frac{1}{64}.$$

Homework: 4.10, 4.11, 4.12, 4.14 on pages 28-29