Section 1.5 Random Variables

Random Experiment: Flip a coin once. $\mathcal{C} = \{c;\ where\ c\ is\ T\ or\ H\}$.

We let $X(c) = x$ be a function such that $X(c)=1$; *if c is H* and $X(c)=0$; *if c is T*.

So, **X** is a function that takes us from the sample space $\mathcal{C}$ to another sample space of real number D where $D=\{x;\ x=0\ or\ x=1\}$.

$\mathcal{C}$		D
H	$X(c=H)$	**1**
T	$X(c=T)$	0

X is a random variable that takes us from the sample space $\mathcal{C}$ to a the new sample space on the real line D. In other words, the random variable **X** assigns numerical values to the experimental outcomes in $\mathcal{C}$.

Defn 8: Consider a random experiment with a sample space $\mathcal{C}$**.** A function **X ,** which assigns to each element $c \in \mathcal{C}$ one and only real number $X(c) = x$, is called a random variable.

If the elements in $\mathcal{C}$ are real number then $\mathcal{C}$ and D are the same; i.e. $\mathcal{C} = D$.

Example: Roll a die once. $\mathcal{C} = \{1,2,3,4,5,6\}$ and $D = \{1,2,3,4,5,6\}$.

We define the $P(d) = P_r(x \in d) = P_x(d)$ to be the probability of the event d .

Note: Since $P(d) = P(c)$ both are probability set functions.

Example: Roll a die twice and **X** be the sum on the faces of the two dies.

X	2	3	4	5	6	7	8	9	10	11	12
$f(x)=P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Note: 1. $f(x) > 0$ and 2. $\sum_{i=1}^{n} P(x_i) = 1$. If both conditions are satisfied, the $f(x)$ is a probability density function(**pdf**) or probability mass function(pmf).

Example 2 page 33: In a lot of One hundred fuses, 20 fuses are defective. If we select five fuses at random, what is the probability that all five are good? Is this a pdf?

Let X = The number of good fuses, then $D = \{x;\ x = 0,1,2,3,4,5\}$. The probability distribution function then is

$$f(x) = P(x) = \begin{cases} \dfrac{\binom{20}{5-x}\binom{80}{x}}{\binom{100}{5}} \end{cases} \quad ; \ for \ \ x = 0,1,2,3,4,5$$. Is it a pdf ?

$$f(x) = P(x) = \begin{cases} \dfrac{\binom{20}{5-0}\binom{80}{0}}{\binom{100}{5}} = 0.000206 \ ; \ X=0 \\ \dfrac{\binom{20}{5-1}\binom{80}{1}}{\binom{100}{5}} = 0.005148 \ ; \ X=1 \\ \dfrac{\binom{20}{5-2}\binom{80}{2}}{\binom{100}{5}} = 0.047849 \ ; \ X=2 \\ \dfrac{\binom{20}{5-3}\binom{80}{3}}{\binom{100}{5}} = 0.207344 \ ; \ X=3 \\ \dfrac{\binom{20}{5-4}\binom{80}{4}}{\binom{100}{5}} = 0.420144 \ ; \ X=4 \\ \dfrac{\binom{20}{5-5}\binom{80}{5}}{\binom{100}{5}} = 0.319309 \ ; \ X=5 \end{cases}$$

P(X=0)+P(X=1) +P(X=2)+P(X=3) P(X=4)+P(X=5) = 1

Note: 1. $f(x) > 0$ and 2. $\sum_{i=1}^{n} P(x_i) = 1$. Yes, $f(x)$ is a pdf .

Example 1 page 31(Handout): Let the random variable X be the number of flips necessary to produce the first head.

$\overbrace{TTTT\ldots T}^{X-1}\ H$ then $f(x) = \left(\tfrac{1}{2}\right)^{x-1}\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^{x}$; $x = 1, 2, 3, \ldots$

$\sum_{x=1}^{\infty}\left(\tfrac{1}{2}\right)^{x} = \sum_{i=0}^{\infty}\tfrac{1}{2}\left(\tfrac{1}{2}\right)^{x}$; geometric series; $a = \tfrac{1}{2}$ *and* $r = \tfrac{1}{2}$. $\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$.

Note: 1. $f(x) > 0$ and 2. $\sum_{i=1}^{n} P(x_i) = 1$. Yes, $f(x)$ is a pdf.

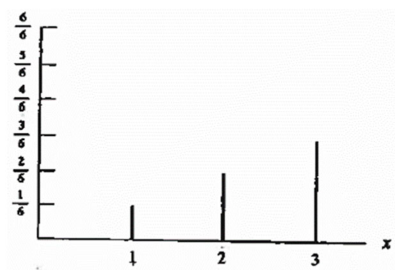
Cumulative Distribution Function

Let $F(x) = P(X \le x)$. $F(x)$ is called the cumulative distribution function(cdf).

For a discrete random variable $F(x) = P(X \le x) = \sum_{w \le x} f(w)$.

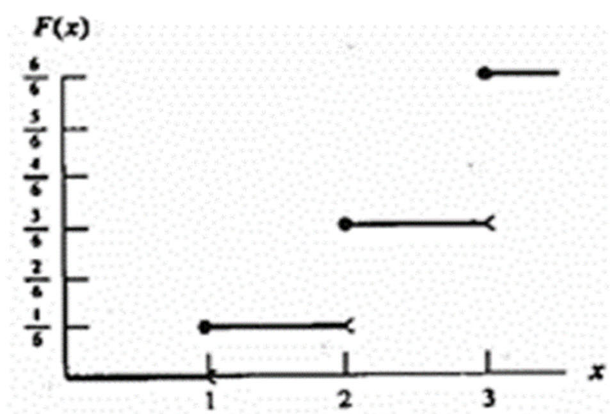
Example: Let $f(x) = \frac{x}{6}$; $x = 1,2,3$. The pdf is given in the table below.

X	$f(x) = P(x)$
1	$\frac{1}{6}$
2	$\frac{2}{6}$
3	$\frac{3}{6}$



The cdf is

$$F(x) = \begin{cases} 0, & x < 1, \\ \frac{1}{6}, & 1 \le x < 2, \\ \frac{3}{6}, & 2 \le x < 3, \\ 1, & 3 \le x. \end{cases}$$



Question: $P(1.5 < x \le 4.5) = P(x = 2) + P(x = 3) = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$ using the pdf.

$P(1.5 < x \le 4.5) = F(4.5) - F(1.5) = 1 - \frac{1}{6} = \frac{5}{6}$ using the cdf.

Note:

1. $0 \le F(x) \le 1$
2. $F(x)$ is an increasing function.
3. $F(y) = 0$ for every point **y** that is less than the smallest value in the space of X.
4. $F(\infty) = 1 \ and \ F(-\infty) = 0$
5. If X is a discrete random variable, then $F(x)$ is a step function and the height of the step at X in the space of X is equal to $f(x) = P(X = x)$.

Homework: 1.48, 1.50, 1.51, 1.54, 1.55(a, b, c only) on pages 35-36 (Handout)