Section 1.6 Random Variables of the Continuous Type

In section 1.5 we learn to find the pdf of $f(x)$ for a **discrete random variable**. Is not as straight forward to come up with the pdf of $f(x)$ for a **continuous random variable**. For a continuous random variable, X, $f(x)$ is a pdf if

1. $f(x) > 0$ (means $f(x)$ is above the x-axis) and 2. $\int_D f(x)\,dx = 1$.

Let $A = \{x;\, a < x < b\}$.

$P(A) = P(a < x < b) = \int_a^b f(x)\,dx$ and $P(X = a) = \int_a^a f(x)\,dx = 0$

Consider the unit circle. Another circle inside the unit circle with a certain radius is generated. If we define the r.v. X to be the radius of the smaller circle, then we have $D = \{x;\, 0 \le x \le 1\}$. What is the pdf of the r.v. X ?

Suggestion: First find the cdf, $F(x)$, of the r.v. X.

$$F(x) = P(X \le x) = \frac{\text{Area of the circle with radius } X}{\text{Area of the unit circle}} = \frac{\pi(x)^2}{\pi(1)^2} = x^2$$

$F(x) = x^2;\, 0 \le x \le 1$. The full description of the cdf is

$$F(x) = \begin{cases} 0 & ;\; x < 0 \\ x^2 & ;\; 0 \le x < 1 \\ 1 & ;\; x \ge 1 \end{cases}$$

Note:

$$F(x) = \int_{w \le x} f(w)\,dw \;;\; \text{since } D = \{x;\, 0 \le x \le 1\}$$

$$F(x) = \int_0^x f(w)\,dw \;;\; x \in D$$

We know that $F(x) = P(X \le x) = \int_0^x f(w)\,dw = x^2$.

From the fundamental theorem of calculus, we know that

$$\frac{d}{dx}\left(F(x)\right) = f(x) \Rightarrow \frac{d}{dx}\left(x^2\right) = f(x) \Rightarrow f(x) = 2x\ ;\ 0 \le x \le 1\ .$$

$f(x) = 2x\ ;\ 0 \le x \le 1$.

Note: $\int_0^1 2x\, dx = \frac{2x^2}{2}\Big|_0^1 = x^2\Big|_0^1 = 1 - 0 = 1$ *and* $f(x) > 0$. Hence $f(x)$ is a pdf since both conditions are satisfied.

Question: $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} 2x\, dx = \frac{2x^2}{2}\Big|_{\frac{1}{4}}^{\frac{1}{2}} = x^2\Big|_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$ using the pdf.

$P\left(\frac{1}{4} < x < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$ using the cdf.

Example: Given the function $f(x) = \frac{1}{40}e^{-\frac{x}{40}}\ ;\ 0 < x < \infty$, (a) show that it's a pdf and (b) Compute $P(40 < x < \infty)$.

Note: (a) $\int_0^\infty \frac{1}{40}e^{-\frac{x}{40}}\, dx = \frac{\frac{1}{40}e^{-\frac{x}{40}}}{-\frac{1}{40}}\Bigg|_0^\infty = -e^{-\frac{x}{40}}\Big|_0^\infty = -\lim_{x\to\infty} e^{-\frac{x}{40}} - \left(-e^{-\frac{0}{40}}\right) = 0 - (-1) = 1$ and $f(x) > 0$. Hence $f(x)$ is a pdf since both conditions are satisfied.

(b) $\int_{40}^\infty \frac{1}{40}e^{-\frac{x}{40}}\, dx = \frac{\frac{1}{40}e^{-\frac{x}{40}}}{-\frac{1}{40}}\Bigg|_{40}^\infty = -e^{-\frac{x}{40}}\Big|_{40}^\infty = -\lim_{x\to\infty} e^{-\frac{x}{40}} - \left(-e^{-\frac{40}{40}}\right) = 0 - \left(-e^{-1}\right) = e^{-1}$

Question: What is $P(0 < x < 40)$?

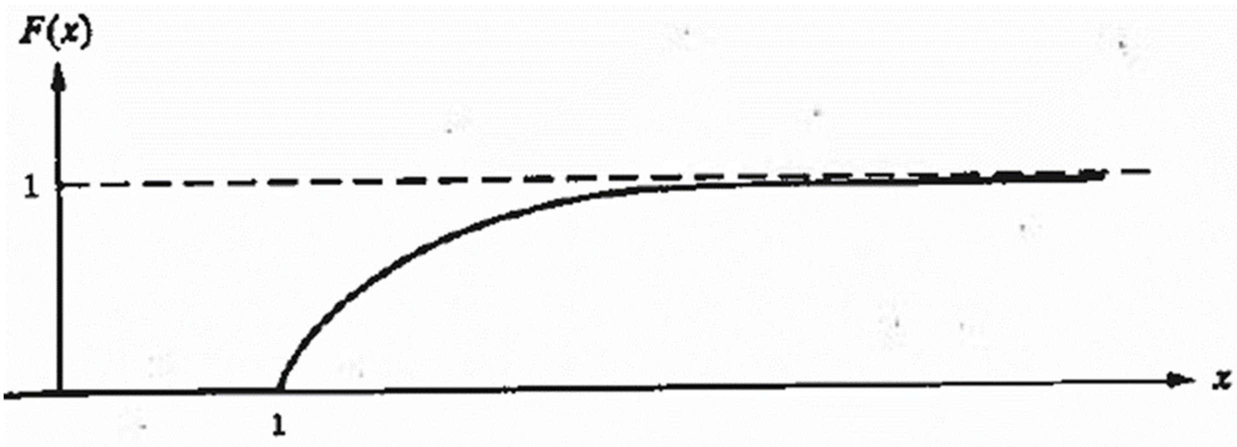
$$P(0 < x < 40) = \int_0^{40} \frac{1}{40}e^{-\frac{x}{40}}\, dx = \frac{\frac{1}{40}e^{-\frac{x}{40}}}{-\frac{1}{40}}\Bigg|_0^{40} = -e^{-\frac{x}{40}}\Big|_0^{40} = -e^{-\frac{40}{40}} - \left(-e^{-\frac{0}{40}}\right) = 1 - e^{-1}$$

Or use the compliment $P(0 < x < 40) = 1 - P(40 < x < \infty) = 1 - e^{-1}$.

Example: consider the pdf $f(x) = \frac{2}{x^3}$; $1 < x < \infty$. What is the cdf ?

$F(x) = P(X \le x) = \int_1^x \frac{2}{w^3}\, dw = \left.\frac{2w^{-2}}{-2}\right|_1^x = -\left.\frac{1}{w^2}\right|_1^x = 1 - \frac{1}{x^2}$. The full description of the cdf is $F(x) = \begin{cases} 0 & ;\ x < 1 \\ 1 - \frac{1}{x^2} & ;\ x \ge 1 \end{cases}$

The graph of the cdf is



Homework: 1.64, 1.66, 1.67, 1.68, 1.69 on pages 43-44(Handout)