Section 1.7 Properties of the Cumulative Distribution Function (cdf)

Cumulative Distribution Function (cdf):

$$F(x) = P(X \le x) = \begin{cases} \sum_{w \le x} f(x) & \textit{for discrete r.v.} \\ \int_{-\infty}^{x} f(x)\,dx & \textit{for continuous r.v.} \end{cases}$$

Note: For Continuous r.v. if $f(x)$ is continuous, so is the $F(x)$. Then $f(x) = F'(x)$.

<u>General Properties of the cdf,</u> $F(x)$

1. $0 \le F(x) \le 1$ since $0 \le P(X \le x) \le 1$.

2. $F(x)$ is a non-decreasing function. It starts at 0 and increases to 1.

3. $F(\infty) = 1$ *and* $F(-\infty) = 0$

   $P(a < x \le b) = F(b) - F(a)$

   $P(x = b) = F(b) - F(b-)$ where $F(b-)$ is the left hand limit. **(For discrete r.v.)**

   $P(x = b) = F(b) - F(b-) = 0$ since always $F(b) = F(b-)$ **for continuous r.v.**

4. If $F(b+) - F(b) = 0$ implies that $F(x)$ is continuous from the right.

<u>Derivation of the Uniform Distribution</u> **-** $f(x) = \frac{1}{b-a}$; $a \le x \le b$

Consider the identity function ; $f(x) = x$ defined on the interval $D = [a, b]$.
Let $d \in D$. The probability of ***d*** is proportional to the length of ***d***. That is if ***d*** is the interval $[a, x]$, $x \le b$, then $P(d) \propto (x-a)$.
$P(d) = P(x \in d) = P(a \le X \le x) = (x-a) \times c$ where ***c*** is the constant of proportionality. **Need to solve for *c*.**

If $x = b$, then $1 = P(a \le X \le x) = P(a \le X \le b) = (b-a) \times c \Rightarrow c = \frac{1}{b-a}$.
Hence, $F(x) = P(X \le x) = P(a \le X \le x) = (x-a) \times c = \frac{x-a}{b-a}$.

$F(x) = \frac{x-a}{b-a};\ a \le x \le b$. The full description of the cdf is

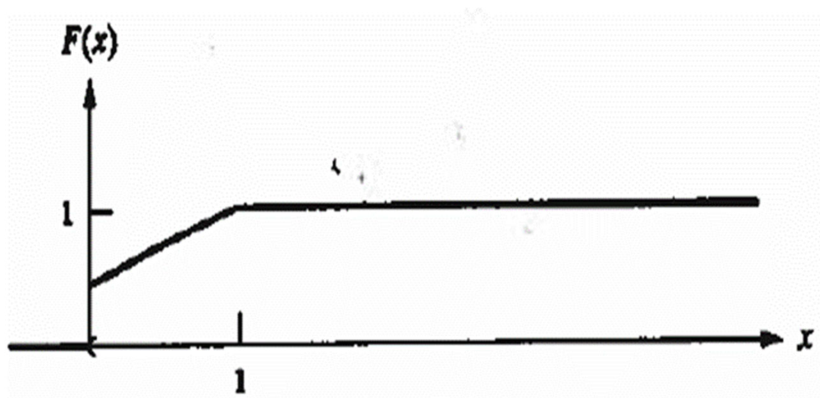
$$F(x) = \begin{cases} 0 & ;\ x < a \\ \frac{x-a}{b-a} & ;\ a \le x \le b \\ 1 & ;\ x > b \end{cases}$$

The pdf of **X** is $f(x) = F'(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a} \Rightarrow f(x) = \frac{1}{b-a};\ a \le x \le b$.

Uniform Probability Distribution: $f(x) = \frac{1}{b-a};\ a \le x \le b$

Example: Now consider the a cdf that is neither of continuous nor of the discrete type.

$$F(x) = \begin{cases} 0 & ;\ x < 0 \\ \frac{x+1}{2} & ;\ 0 \le x < 1 \\ 1 & ;\ x \ge 1 \end{cases}$$



Question: **what is the pdf of X?** Please note that this is a mixed pdf. It has both discrete and continuous parts.

At $x = 0$ there is jump which it makes it discrete.

$P(x=0) = F(0) - F(0-) = \frac{0+1}{2} - \frac{0+0}{2} = \frac{1}{2}$; *i.e.* $P(x=0) = \frac{1}{2}$

In the interval $0 < x < 1$ the function is continuous.

$f(x) = F'(x) = \frac{d}{dx}\left(\frac{x+1}{2}\right) = \frac{1}{2} \Rightarrow f(x) = \frac{1}{2};\ 0 < x < 1$

$$f(x) = \begin{cases} \frac{1}{2} & ;\ x = 0 \\ \frac{1}{2} & ;\ 0 < x < 1 \end{cases}$$

(a) $P(3 \le x \le 6) = F(6) - F(3) = 1 - 1 = 0$

(b) $P\left(-3 < x \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F(-3) = \frac{\frac{1}{2}+1}{2} - 0 = \frac{3}{4} - 0 = \frac{3}{4}$

(c) $P\left(x = \frac{1}{4}\right) = F\left(\frac{1}{4}\right) - F\left(\frac{1}{4}-\right) = \frac{\frac{1}{4}+1}{2} - \frac{\frac{1}{4}+1}{2} = \frac{5}{8} - \frac{5}{8} = 0$. It has to be 0 since it's continuous at $x = \frac{1}{4}$.

(d) $P(x=0) = F(0) - F(0-) = \frac{0+1}{2} - 0 = \frac{1}{2}$. The probability exists since it's discrete at $x = 0$.

Example: Consider the r.v. X with pdf $f(x) = \frac{1}{2};\ -1 < x < 1,\ zero\ elsewhere$. Define another r.v. $Y = u(x) = x^2$. Find the c.d.f. and p.d.f. of Y .

What are the limits for Y? Since

$Y = x^2 \Rightarrow when\ x = 1;\ Y = (1)^2 = 1\ and\ when\ x = -1;\ Y = (-1)^2 = 1$. Therefor $0 < y < 1$.

$$G(Y) = P(Y \le y) = P\left(X^2 \le y\right) = P\left(-\sqrt{y} \le X \le \sqrt{y}\right)$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\ dx = \frac{x}{2}\Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{\sqrt{y}}{2} - \left(-\frac{\sqrt{y}}{2}\right) = \frac{2\sqrt{y}}{2} = \sqrt{y}\ ;\ 0 < y < 1$$

$$G(Y) = \begin{cases} 0 & ;\quad y \le 0 \\ \sqrt{y} & ;\quad 0 < y < 1 \\ 1 & ;\quad y \ge 1 \end{cases}$$

$$\text{Since } g(y) = G'(y) = \frac{d}{dx}\left(\sqrt{y}\right) = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$$

$$g(y) = \begin{cases} \frac{1}{2\sqrt{y}} & ;\quad 0 < y < 1 \\ 0 & ;\quad else\ where \end{cases}$$

Homework: 1.71, 1.72, 1.73, 1.74, 1.75, 1.76 on pages 51(Handout)