Section 1.8 Expectation of Random Variables

$E(X)$ is the called "the expected value of X" and $E(X) = \mu$

$$E(X) = \begin{cases} \sum_{all\,X} x\,f(x) = \sum_{all\,X} x\,P(x) \;; & \textit{for discrete r.v.} \\ \int_{-\infty}^{\infty} x\,f(x)\,dx \;; & \textit{for continuous r.v.} \end{cases}$$

Example: Roll two dice. Let X = "sum of the two numbers on the face of the two dice". Find the pdf X and then compute the E(X). **The pdf was done in section 1.5 .**

x	P(x)	xP(x)
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	20/36
6	5/36	30/36
7	6/36	42/36
8	5/36	40/36
9	4/36	36/36
10	3/36	30/36
11	2/36	22/36
12	1/36	12/36

$E(X) = \sum_{x=2}^{12} xP(x) = \frac{252}{36} = 7$. Remember $E(X) = \mu = 7$

Example: Let $f(x) = 4x^3 \;;\; 0 < x < 1.$

$$\textit{The } E(X) = \mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 x\left(4x^3\right)dx = \frac{4x^5}{5}\Big|_0^1 = \frac{4}{5}$$

Note: If we have

$E(X) = \begin{cases} \sum_{all\,X} x\,f(x) \\ \int_{-\infty}^{\infty} x\,f(x)\,dx \end{cases}$ and define $Y = u(x)$ then $E(Y) = E(u(x)) = \begin{cases} \sum_{all\,X} u(x)\,f(x) \\ \int_{-\infty}^{\infty} u(x)\,f(x)\,dx \end{cases}$

In general:

a. $E(k) = k$ where k is a constant

b. $E(kX) = k\,E(X)$

c. $E(k_1X + k_2Y) = E(k_1X) + E(k_2Y) = k_1E(X) + k_2E(Y)$

d. If $E(k_1X_1 + k_2X_2 + \ldots + k_nX_n) = k_1E(X_1) + k_2E(X_2) + \ldots + k_nE(X_n)$

Example 1.8.4: $f(x) = 2(1-x)$; $0 < x < 1$.

$$E(X) = \int_0^1 x\left[2(1-x)\right]\, dx = \int_0^1 2x - 2x^2\, dx = \tfrac{2x^2}{2} - \tfrac{2x^3}{3}\Big|_0^1 = \left(1-\tfrac{2}{3}\right) - (0) = \tfrac{1}{3}$$

$$E(X^2) = \int_0^1 x^2\left[2(1-x)\right]\, dx = \int_0^1 2x^2 - 2x^3\, dx = \tfrac{2x^3}{3} - \tfrac{2x^4}{4}\Big|_0^1 = \left(\tfrac{2}{3}-\tfrac{2}{4}\right) - (0) = \tfrac{1}{6}$$

$$E(6X + 3X^2) = 6E(X) + 3E(X^2) = 6\left(\tfrac{1}{3}\right) + 3\left(\tfrac{1}{6}\right) = 2 + \tfrac{1}{2} = \tfrac{5}{2} \text{ or } 2.5$$

Example 1.8.5: $f(x) = \frac{x}{6}$; $x = 1, 2, 3$

$$E(X) = \sum_{x=1}^{3} x\tfrac{x}{6} = 1\times\tfrac{1}{6} + 2\times\tfrac{2}{6} + 3\times\tfrac{3}{6} = \tfrac{1}{6} + \tfrac{4}{6} + \tfrac{9}{6} = \tfrac{14}{6} = \tfrac{7}{3}$$

$$E(X^2) = \sum_{x=1}^{3} x^2\tfrac{x}{6} = (1)^2\times\tfrac{1}{6} + (2)^2\times\tfrac{2}{6} + (3)^2\times\tfrac{3}{6} = \tfrac{1}{6} + \tfrac{8}{6} + \tfrac{27}{6} = \tfrac{36}{6} = 6$$

$$E(X^3) = \sum_{x=1}^{3} x^3\tfrac{x}{6} = (1)^3\times\tfrac{1}{6} + (2)^3\times\tfrac{2}{6} + (3)^3\times\tfrac{3}{6} = \tfrac{1}{6} + \tfrac{16}{6} + \tfrac{81}{6} = \tfrac{98}{6} = \tfrac{49}{3}$$

x	P(x)	xP(x)	x^2 P(x)	X^3 P(x)
1	1/6	1/6	1/6	1/6
2	2/6	4/6	8/6	16/6
3	3/6	9/6	27/6	81/6
		$E(X) = \frac{14}{6} = \frac{7}{3}$	$E(X^2) = \frac{36}{6} = 6$	$E(X^3) = \frac{98}{6} = \frac{49}{3}$

Example 1.8.6: $f(x) = \frac{1}{5}$; $0 < x < 5$.

$$E(X) = \int_0^5 x\left(\tfrac{1}{5}\right)dx = \tfrac{x^2}{10}\Big|_0^5 = \tfrac{25}{10} = \tfrac{5}{2}$$

$$E(X^2) = \int_0^5 x^2\left(\tfrac{1}{5}\right)dx = \tfrac{x^3}{15}\Big|_0^5 = \tfrac{125}{15} = \tfrac{25}{3}$$

$E(5 - X) = E(5) - E(X) = 5 - \frac{5}{2} = \frac{5}{2}$ and

$$E(X(5-X)) = E(5X - X^2) = 5E(X) - E(X^2) = 5\left(\tfrac{5}{2}\right) - \tfrac{25}{3} = \tfrac{25}{2} - \tfrac{25}{3} = \tfrac{75}{6} - \tfrac{50}{6} = \tfrac{25}{6}$$

Observation: $E(X(5-X)) = \frac{25}{6} \neq \frac{25}{4} = \left(\frac{5}{2}\right)\left(\frac{5}{2}\right) = E(X)E(5-X)$

In general, $E(X(5-X)) \neq E(X)E(5-X)$. **They are equal only if the random variables are independent.**

Example 1.8.4: A bowl contains five chips, which cannot be distinguished by a sense of touch alone. Three of the chips are marked \$1 each and the remaining two are marked \$4 each. A player is blindfolded and draws at random and without replacement, two chips from the bowl. The player is paid an amount equal to the sum of the values of the two chips that he draws and the game is over. If it cost \$4.75 to play the game, would we care to participate for any protracted period of time? To answer the question, we need to compute the expected value, E(X), of the game.

Let the r.v. X be the number of chips that are marked \$1.

Define $r = \binom{3}{x}\binom{2}{2-x}$; *where* $x = 0,\ 1,\ 2$ and $k = \binom{5}{2}$. $f(x) = P(X) = \frac{r}{k}$

The pdf of the r.v. X is $f(x) = \dfrac{\binom{3}{x}\binom{2}{2-x}}{\binom{5}{2}}$; $x = 0,\ 1,\ 2$.

$$f(x)=\begin{cases} \dfrac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}}=\dfrac{1\times 1}{\frac{5!}{2!\times 3!}}=\dfrac{1}{\frac{5\times 4\times 3!}{2!\times 3!}}=\dfrac{1}{10}; & X=0 \\ \dfrac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}}=\dfrac{3\times 2}{\frac{5!}{2!\times 3!}}=\dfrac{6}{\frac{5\times 4\times 3!}{2!\times 3!}}=\dfrac{6}{10}; & X=1 \\ \dfrac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}}=\dfrac{3\times 1}{\frac{5!}{2!\times 3!}}=\dfrac{3}{\frac{5\times 4\times 3!}{2!\times 3!}}=\dfrac{3}{10}; & X=2 \end{cases}$$

$E(X)=0\times\frac{1}{10}+1\times\frac{6}{10}+2\times\frac{3}{10}=\frac{12}{10}=1.2$

Let $u(X)$ be the profit. $u(X)=1(X)+4(2-X)=8-3X$

$E(u(X))=E(8-3X)=8-3E(X)=8-3(1.2)=8-3.6=4.4$

OR Let the r.v. X be the number of chips that are marked \$4.

Define $r=\binom{2}{x}\binom{3}{2-x}$; *where* $x=0,\ 1,\ 2$ and $k=\binom{5}{2}$. The $f(x)=P(X)=\frac{r}{k}$

The pdf of the r.v. X is $f(x)=\dfrac{\binom{2}{x}\binom{3}{2-x}}{\binom{5}{2}}$; $x=0,\ 1,\ 2$.

$$f(x)=\begin{cases}\dfrac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}}=\dfrac{1\times 3}{\frac{5!}{2!\times 3!}}=\dfrac{3}{\frac{5\times 4\times 3!}{2!\times 3!}}=\dfrac{3}{10}; & X=0\\[2ex] \dfrac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}}=\dfrac{2\times 3}{\frac{5!}{2!\times 3!}}=\dfrac{6}{\frac{5\times 4\times 3!}{2!\times 3!}}=\dfrac{6}{10}; & X=1\\[2ex] \dfrac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}}=\dfrac{1\times 1}{\frac{5!}{2!\times 3!}}=\dfrac{1}{\frac{5\times 4\times 3!}{2!\times 3!}}=\dfrac{1}{10}; & X=2\end{cases}$$

$$E(X)=0\times\tfrac{3}{10}+1\times\tfrac{6}{10}+2\times\tfrac{1}{10}=\tfrac{8}{10}=0.8$$

Let $u(X)$ be the profit. $u(X)=4(X)+1(2-X)=4X+2-X=3X+2$

$$E(u(X))=E(3X+2)=3E(X)+2=3(0.8)+2=2.4+2=4.4$$

No, we should not play this game. We pay \$4.75 to play this game but we are expected to make only \$4.4. We will be losing (4.75-4.4=0.35) \$0.35 every time we play the game. If we play the game 100 times we are expected to lose (100x0.35=35) \$35.

Homework: 8.2, 8.3, 8.5, 8.6, 8.7, 8.8, 8.11 on pages 56-57