Section 1.9 Some Special Mathematical Expectations

Let X be a random variable.

1. $\mu = E(X)$: Expected value of X or the mean value of X
2. $\sigma^2 = E(X-\mu)^2$: The mean squared deviations from the mean μ or the variance of X
3. $\sigma = \sqrt{\sigma^2}$: The standard deviation of X

Note: $\sigma^2 = E(X-\mu)^2 = E\left(X^2 - 2X\mu - \mu^2\right) = E\left(X^2\right) - 2\mu E(X) + \mu^2 = E\left(X^2\right) - \mu^2$

4. Moment-generating function---mgf

Definition: Let X be a random variable and h a positive number.

$$E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \ \ (\textit{for continuous} \text{ r.v.}) \ \ \textit{or} \ \ \sum_{all\ x} e^{tx} f(x) \ \ (\textit{for discrete} \text{ r.v.})$$

exists in $-h < t < h$, then this is called the moment generating function (mgf).

Note: (a) Not every distribution has an mgf. If it exists, it's unique. It has a one-to-one relationship with the pdf. (b) Given $f(x)$, we may be able to calculate the mgf. However, given the mgf say $M(t)$ it may not be obvious how to find $f(x)$. (c) Since $M(t)$ exits in $-h < t < h$ all derivatives exist at $t = 0$; i.e.

$$\frac{d}{dt}\left[E\left(e^{tX}\right)\right] = \frac{d}{dt}\left[\int_{-\infty}^{\infty} e^{tx} f(x)\, dx\right] = \int_{-\infty}^{\infty} \frac{d}{dt}\left[e^{tx} f(x)\right] dx.$$

Example 1. Given $M(t) = \frac{1}{10}e^{t} + \frac{2}{10}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t}$ for all real values of t, what is the p.d.f. of X ?

Since $E\left(e^{tX}\right) = \sum_{all\ x} e^{tx} f(x)$ we can see that $f(x) = \frac{x}{10}$; $\textit{for}\ \ x = 1,2,3,4$.

Example 2. Given $M(t)=\frac{1}{(1-t)^2}$; *where* $t<1$, what is the p.d.f. of X ?

In this case, it's not obvious what the p.d.f. of X is. Consider the function $f(x)=xe^{-x}$; $0<x<\infty$ and find it's mgf.

$E(e^{tX})=\int_{-\infty}^{\infty} e^{tx} f(x)\,dx=\int_0^{\infty} e^{tx} x e^{-x}\,dx=\int_0^{\infty} x e^{-x+tx}\,dx=\int_0^{\infty} x e^{-x(1-t)}\,dx$. Use the Gamma Function to evaluate this integral. $\int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}}\,dx=\Gamma(\alpha)\beta^{\alpha}$ where $\Gamma(\alpha)=(\alpha-1)!$

$\int_0^{\infty} x e^{-x(1-t)}\,dx=\int_0^{\infty} x^{2-1} e^{-\frac{x}{\frac{1}{(1-t)}}}\,dx$; Hence $\alpha=2$ *and* $\beta=\frac{1}{(1-t)}$

$\int_0^{\infty} x e^{-x(1-t)}\,dx=\Gamma(2)\left(\frac{1}{1-t}\right)^2=(2-1)!\left(\frac{1}{1-t}\right)^2=\left(\frac{1}{1-t}\right)^2$; $M(t)=\left(\frac{1}{1-t}\right)^2$; $t<1$

Properties:

(a) Since all the derivatives exists at $t=0$; $M(t=0)=E(e^{0x})=E(1)=1$.

(b) $\frac{d}{dt}[M(t)]=M'(t)=\left[\int_{-\infty}^{\infty}\frac{d}{dt}(e^{tx}f(x))\,dx\right]=\int_{-\infty}^{\infty} xe^{tx} f(x)dx$

$\Rightarrow M'(t=0)=\int_{-\infty}^{\infty} xe^{0x} f(x)dx \quad \Rightarrow M'(t=0)=\int_{-\infty}^{\infty} xf(x)dx$;

$$Hence,\ M'(t=0)=E(X)=\mu$$

Similarly for a discrete random variable :

$\frac{d}{dt}[M(t)]=M'(t)=\sum_{all\,x} e^{tx} f(x)=\sum_{all\,x} xe^{tx} f(x);\ for\ t=0\ =\sum_{all\,x} xf(x)=E(X)$

$$M''(t)=\begin{cases}\int_{-\infty}^{\infty}\frac{d}{dt^2}(e^{tx}f(x))\,dx=\int_{-\infty}^{\infty} x^2e^{tx}f(x)dx;\ for\ t=0\ \int_{-\infty}^{\infty} x^2f(x)dx=E(X^2)\\ \sum_{all\,x}\frac{d}{dt^2}(e^{tx}f(x))=\sum_{all\,x}\frac{d}{dt^2}(x^2e^{tx}f(x));\ for\ t=0\ \sum_{all\,x} x^2f(x)=E(X^2)\end{cases}$$

Note: $M'(0)=[E(X)]=\mu$ *and* $M''(0)=E(X^2)$
and $\sigma^2=E(X^2)-[E(X)]^2=M''(0)-[M'(0)]^2$

Example 3: Let $f(x) = xe^{-x}$; $0 < x < \infty$.

Using the definition of the Expected Value evaluate the following:

$$E(X) = \int_0^\infty xxe^{-x}\, dx = \int_0^\infty x^2 e^{-x}\, dx = \Gamma(3)(1)^3 = 2$$

$$E(X^2) = \int_0^\infty x^2 xe^{-x}\, dx = \int_0^\infty x^3 e^{-x}\, dx = \Gamma(4)(1)^4 = 6$$

$$\sigma^2 = E(X^2) - [E(X)]^2 = 6 - (2)^2 = 2.$$

Note: The mgf of $f(x) = xe^{-x}$; $0 < x < \infty$ **is** $M(t) = \frac{1}{(1-t)^2}$; *where* $t < 1$.

Use it's mgf, $M(t) = \frac{1}{(1-t)^2}$, to evaluate the $E(X)$, $E(X^2)$, and σ^2.

$$E(X) = M'(t) = -2(1-t)^{-3}(-1) \Rightarrow M'(0) = -2(1-0)^{-3}(-1) = 2$$

$$E(X^2) = M''(t) = 2(-3)(1-t)^{-4}(-1) \Rightarrow M''(0) = 6(1-0)^{-4} = 6$$

$$\sigma^2 = E(X^2) - [E(X)]^2 = 6 - (2)^2 = 2$$

In general if **m** is the m^th^ derivative then $E(X^m)$ is the m^th^ moment of X.

$$M^m(0) = \begin{cases} \int_{-\infty}^{\infty} x^m e^{tx} f(x)\, dx = E(X^m) & \text{for } t = 0 \ (\text{continuous}) \\ \sum_{all\ x} x^m e^{tx} f(x) = E(X^m) & \text{for } t = 0 \ (\text{discrete}) \end{cases}$$

Sometimes a moment may not exist.

Example 4: Let $f(x) = \frac{1}{x^2}$; $1 < x < \infty$.

$E(X) = \int_1^\infty x\frac{1}{x^2}\, dx = \int_1^\infty \frac{1}{x}\, dx = \ln x\Big|_1^\infty = Undefined$; Hence the first moment does not exist.

Example 5: Given the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ *converges to* $\frac{\pi^2}{6}$, does the mgf of $f(x)=\frac{6}{\pi^2 x^2}$; *for* $x=1,2,3,4,\ldots$ exists?

$M(t)=\sum_{x=1}^{\infty}\frac{6e^{tx}}{\pi^2 x^2}$; Apply the Ratio Test to find if this series converges.

Ratio Test $\lim_{x\to\infty}\frac{\frac{6e^{t(x+1)}}{\pi^2(x+1)^2}}{\frac{6e^{tx}}{\pi^2 x^2}}=\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^2 e^{t(x+1)}e^{-tx}=\lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right)^2 e^t=e^t$.

Note: If $e^t>1$ the series diverges
If $e^t<1$ the series converges
If $e^t=1$ inconclusive

Solve for t : $e^t<1 \Rightarrow \ln e^t<\ln 1 \Rightarrow t<0$

There is no positive number h such that $-h<t<h$. Therefore the mgf, $M(t)$, does not exist.

Homework: 1.9.1, 1.9.2, 1.9.6, 1.9.7, 1.9.18, 1.9.23 ; p.p. 64-67