Chapter 2-Section 2.1

As the one dimensional random variable took us from the $\mathcal{C}$ to $\mathcal{A}$ sample space so are the two random variables taking us from the two dimensional space $\mathcal{C}$ to a dimensional space $\mathcal{A}$.

$$P(A)=P((x,y)\in A)=\begin{cases}\iint\limits_{all\ A} f(x,y)\,dx\,dy & \to continuous\\ \sum\sum\limits_{all\ A} f(x,y) & \to discrete\end{cases}$$

For $f(x,y)$ to be a pdf we need

1. $f(x,y)>0$

2. $\sum\limits_{x}\sum\limits_{y} f(x,y)=1$ *for discrete* and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$ *for continuous*.

Example 1: $f(x,y)=\dfrac{9}{4^{x+y}}$; $x=1,2,3,\ldots$ *and* $y=1,2,3,\ldots$

1. $f(x,y)>0$

2.
$$\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\frac{9}{4^{x+y}}=9\left[\left(\sum_{x=1}^{\infty}\frac{1}{4^x}\right)\left(\sum_{y=1}^{\infty}\frac{1}{4^y}\right)\right]=9\left[\left(\sum_{x=0}^{\infty}\tfrac{1}{4}\left(\tfrac{1}{4}\right)^x\right)\left(\sum_{y=0}^{\infty}\tfrac{1}{4}\left(\tfrac{1}{4}\right)^y\right)\right]$$
$$=9\left[\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right)\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right)\right]=9\left(\frac{1}{3}\times\frac{1}{3}\right)=1$$

Example 2: $f(x,y)=4xye^{-x^2}e^{-y^2}$; $0<x<\infty$ *and* $0<y<\infty$

1. $f(x,y)>0$

2.
$$\int_0^{\infty}\int_0^{\infty} 4xye^{-x^2}e^{-y^2}\,dx\,dy=\int_0^{\infty} 2xe^{-x^2}dx\int_0^{\infty} 2ye^{-y^2}dy$$
$$=\int_0^{\infty} e^{-u}du\int_0^{\infty} e^{-u}du=1\times1=1$$

Given the joint pdf $f(x,y)$, the marginal pdfs of X and Y are :

$$f(x)=\begin{cases}\int_{all\ y} f(x,y)\,dy & \rightarrow continuous \\ \sum_{all\ y} f(x,y) & \rightarrow discrete\end{cases}$$

and

$$f(y)=\begin{cases}\int_{all\ x} f(x,y)\,dx \\ \sum_{all\ x} f(x,y)\end{cases}$$

Example 3: Marginal pdfs for a discrete random variable.

		Y			
		1	2	3	$f(x)$
X	1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
	2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{6}{10}$
$f(y)$		$\frac{2}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	

a. Find the marginal pdfs of X and Y.

X	$f(x)$		Y	$f(y)$
1	$\frac{4}{10}$		1	$\frac{2}{10}$
2	$\frac{6}{10}$		2	$\frac{3}{10}$
			3	$\frac{5}{10}$

b. Compute the $P(x=1,\ y=3)$ *and* $P(x=2,\ y\geq 2)$.

$P(x=1,\ y=3)=\frac{2}{10}$ *and* $P(x=2,\ y\geq 2)=P(x=2,\ y=2)+P(x=2,\ y=3)=\frac{2}{10}+\frac{3}{10}=\frac{5}{10}=\frac{1}{2}$

Expectation of $u(x,y)$ is $E(u(x,y))=\begin{cases}\sum_{all\ x}\sum_{all\ y} u(x,y)\,f(x,y) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y)\,f(x,y)\,dx\,dy\end{cases}$.

Mgf : $E\left(e^{t_1x+t_2y}\right)=\begin{cases}\sum_{all\ x}\sum_{all\ y} e^{t_1x+t_2y} f(x,y) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dx\,dy\end{cases}$.

c. Compute the $E(XY)$.

$$E(XY)=\sum_{x=1}^{2}\sum_{y=1}^{3}(x\times y)f(x,y)$$
$$=(1\times1)\tfrac{1}{10}+(1\times2)\tfrac{1}{10}+(1\times3)\tfrac{2}{10}+(2\times1)\tfrac{1}{10}+(2\times2)\tfrac{2}{10}+(2\times3)\tfrac{3}{10}$$
$$=(1)\tfrac{1}{10}+(2)\tfrac{1}{10}+(3)\tfrac{2}{10}+(2)\tfrac{1}{10}+(4)\tfrac{2}{10}+(6)\tfrac{3}{10}=\tfrac{1+2+6+2+8+18}{10}=\tfrac{37}{10}$$

d. Compute the joint mgf, $M(t_1, t_2)$.

$$M(t_1,t_2)=E\left(e^{t_1x+t_2y}\right)=\sum_{x=1}^{2}\sum_{y=1}^{3}e^{xt_1+yt_2}f(x,y)$$
$$=\left(e^{t_1+t_2}\right)\tfrac{1}{10}+\left(e^{t_1+2t_2}\right)\tfrac{1}{10}+\left(e^{t_1+3t_2}\right)\tfrac{2}{10}+\left(e^{2t_1+t_2}\right)\tfrac{1}{10}+\left(e^{2t_1+2t_2}\right)\tfrac{2}{10}+\left(e^{2t_1+3t_2}\right)\tfrac{3}{10}$$
$$=\tfrac{1}{10}e^{t_1+t_2}+\tfrac{1}{10}e^{t_1+2t_2}+\tfrac{2}{10}e^{t_1+3t_2}+\tfrac{1}{10}e^{2t_1+t_2}+\tfrac{2}{10}e^{2t_1+2t_2}+\tfrac{3}{10}e^{2t_1+3t_2}$$

e. Compute the $E(XY)$ using the joint mgf.

$$E(XY)=\frac{\partial^2M(t_1,t_2)}{\partial t_1{}^1\partial t_2{}^1}=M''(t_1=0,\,t_2=0)=(1\times1)\tfrac{1}{10}e^{t_1+t_2}+(1\times2)\tfrac{1}{10}e^{t_1+2t_2}+(1\times3)\tfrac{2}{10}e^{t_1+3t_2}$$
$$+(2\times1)\tfrac{1}{10}e^{2t_1+1t_2}+(2\times2)\tfrac{2}{10}e^{2t_1+2t_2}+(2\times3)\tfrac{3}{10}e^{2t_1+3t_2}$$
$$M''(t_1=0,\,t_2=0)=(1\times1)\tfrac{1}{10}+(1\times2)\tfrac{1}{10}+(1\times3)\tfrac{2}{10}+(2\times1)\tfrac{1}{10}+(2\times2)\tfrac{2}{10}+(2\times3)\tfrac{3}{10}$$
$$E(XY)=\tfrac{1}{10}+\tfrac{2}{10}+\tfrac{6}{10}+\tfrac{2}{10}+\tfrac{8}{10}+\tfrac{18}{10}=\tfrac{37}{10}$$

f. Find the marginal mgfs for X and Y.

$$M(t_1)=M(t_1,\,t_2=0)=\tfrac{1}{10}e^{t_1+0}+\tfrac{1}{10}e^{t_1+2\times0}+\tfrac{2}{10}e^{t_1+3\times0}+\tfrac{1}{10}e^{2t_1+0}+\tfrac{2}{10}e^{2t_1+2\times0}+\tfrac{3}{10}e^{2t_1+3\times0}$$
$$=\tfrac{1}{10}e^{t_1}+\tfrac{1}{10}e^{t_1}+\tfrac{2}{10}e^{t_1}+\tfrac{1}{10}e^{2t_1}+\tfrac{2}{10}e^{2t_1}+\tfrac{3}{10}e^{2t_1}=\tfrac{4}{10}e^{t_1}+\tfrac{6}{10}e^{2t_1}$$
$$mgf\ of\ \ X:\ M(t_1)=\tfrac{4}{10}e^{t_1}+\tfrac{6}{10}e^{2t_1}$$

$$M(t_2)=M(t_1=0,\,t_2)=\tfrac{1}{10}e^{0+t_2}+\tfrac{1}{10}e^{0+2t_2}+\tfrac{2}{10}e^{0+3t_2}+\tfrac{1}{10}e^{2\times0+t_2}+\tfrac{2}{10}e^{2\times0+2t_2}+\tfrac{3}{10}e^{2\times0+3t_2}$$
$$=\tfrac{1}{10}e^{t_2}+\tfrac{1}{10}e^{2t_2}+\tfrac{2}{10}e^{3t_2}+\tfrac{1}{10}e^{t_2}+\tfrac{2}{10}e^{2t_2}+\tfrac{3}{10}e^{3t_2}=\tfrac{2}{10}e^{t_2}+\tfrac{3}{10}e^{2t_2}+\tfrac{5}{10}e^{3t_2}$$
$$mgf\ of\ \ Y:\ M(t_2)=\tfrac{2}{10}e^{t_2}+\tfrac{3}{10}e^{2t_2}+\tfrac{5}{10}e^{3t_2}$$

Example 4: Given the joint pdf $f(x,y) = x + y\ ;\ 0 < x < 1\ \ and\ \ 0 < y < 1$, find the marginal pdfs of X and Y and then compute $P(X \le \frac{1}{2})$ and $P(X + Y \le 1)$:

$$f(x) = \int_0^1 x + y\ dy = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2} \quad \text{Hence, } f(x) = x + \frac{1}{2}\ ;\ 0 < x < 1.$$

$$f(y) = \int_0^1 x + y\ dx = \frac{x^2}{2} + yx\Big|_0^1 = y + \frac{1}{2} \quad \text{Hence, } f(y) = y + \frac{1}{2}\ ;\ 0 < y < 1.$$

$$P(x \le \tfrac{1}{2}) = \int_0^{\frac{1}{2}} f(x)\ dx = \int_0^{\frac{1}{2}} x + \tfrac{1}{2}\ dx = \frac{x^2}{2} + \frac{x}{2}\Big|_0^{\frac{1}{2}} = \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})}{2} = \tfrac{1}{8} + \tfrac{1}{4} = \tfrac{3}{8}$$

$$P(x + y \le 1) = P(y \le 1 - x) = \int_0^1 \int_0^{1-x} x + y\ dy\,dx = \int_0^1 xy + \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= \int_0^1 x(1-x) + \tfrac{(1-x)^2}{2}\ dx = \int_0^1 \tfrac{1}{2} - \tfrac{x^2}{2}\ dx = \frac{x}{2} - \frac{x^3}{6}\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{6} = \tfrac{1}{3}$$

Example 5. Consider the function $f(x,y) = 6x^2 y\ ;\ \ 0 < x < 1\ \ and\ \ 0 < y < 1$.

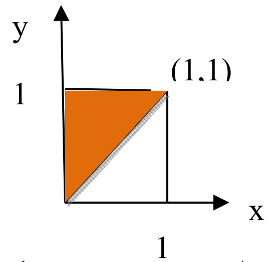
$$P(0 < x < \tfrac{3}{4}, \tfrac{1}{3} < y < 1) = \int_{\frac{1}{3}}^1 \int_0^{\frac{3}{4}} 6x^2 y\,dx\ dy = \int_{\frac{1}{3}}^1 6\tfrac{x^3}{3}\Big|_0^{\frac{3}{4}} y\,dy = \int_{\frac{1}{3}}^1 2\tfrac{27}{64} y\,dy$$

$$= 2\tfrac{27}{64}\tfrac{y^2}{2}\Big|_{\frac{1}{3}}^1 = \tfrac{27}{64}\left(1 - \tfrac{1}{9}\right) = \tfrac{27}{64}\tfrac{8}{9} = \tfrac{3}{8}$$

$$P(x + y \le 1) = P(y \le 1 - x) = \int_0^1 \int_0^{1-x} 6x^2\ y\,dy\,dx = \int_0^1 6x^2\ \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= \int_0^1 3x^2(1-x)^2\ dx = \int_0^1 3x^2\left(1 - 2x + x^2\right)dx = \int_0^1 3x^2 - 6x^3 + 3x^4\ dx = = x^3 - \tfrac{6}{4}x^4 + \frac{3x^5}{5}\Big|_0^1 = 1 - \tfrac{3}{2} + \tfrac{3}{5} = \tfrac{1}{10}$$

Example 6: $f(x,y)=8xy$; $\quad 0<x<y<1$.



$$E(xy)=\int_0^1\int_0^y xy(8xy)\,dxdy=\int_0^1\int_0^y 8x^2y^2\,dxdy=\int_0^1 \tfrac{8}{3}y^2x^3\Big|_0^y dy=\int_0^1 \tfrac{8}{3}y^5dy=\tfrac{8}{18}y^6\Big|_0^1=\tfrac{4}{9}$$

$f(x)=\int_x^1 8xy\,dy=8x\frac{y^2}{2}\Big|_x^1=4x-4x^3$; i.e. $f(x)=4x-4x^3$; $0<x<1$.

$f(y)=\int_0^y 8xy\,dx=8y\frac{x^2}{2}\Big|_0^y=4y^3$; i.e. $f(y)=4y^3$; $0<y<1$.

The cumulative distribution function (cdf) of $f(x,y)$ is

$F(x,y)=P(X\le x\,,Y\le y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy$.

Note: The joint pdf $f(x,y)=\dfrac{\partial^2 F(x,y)}{\partial x^1\partial y^1}$.

Example 7: Consider the joint pdf $f(x,y)=1$; $\quad 0<x<1$ *and* $0<y<1$. Let $Z=X+Y$. Find the pdf and cdf of Z.

$G(Z)=P(Z\le z)=P(X+Y\le z)=P(Y\le z-X)$

$$G(Z)=\begin{cases}0 & ;\ z\le 0\\ \int_0^z\int_0^{z-x} 1\,dy\,dx=\frac{z^2}{2};\ 0<z<1\\ 1-\int_{z-1}^{1}\int_{z-x}^{1} 1\,dy\,dx=1-\frac{(2-z)^2}{2};\ 1\le z<2\\ 1 & ;\ z\ge 2\end{cases}$$

and

$$g(Z)=\begin{cases}z & ;\ 0<z<1\\ 2-z & ;\ 1\le z<2\\ 0 & ;\ e.w.\end{cases}$$

Moment generating Function

Given the joint pdf of X and Y; $f(x,y) = \frac{1}{6}x^3 e^{-y}$; $0 < x < y < \infty$, compute the joint mgf of X and Y; $M(t_1,t_2)$, and the marginal mgfs of X and Y.

$$M(t_1,t_2) = E\left(e^{t_1 x + t_2 y}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y)\,dx\,dy$$

$$M(t_1,t_2) = \int_0^\infty \int_x^\infty e^{t_1 x + t_2 y}\,\tfrac{1}{6}x^3 e^{-y}\,dy\,dx = \tfrac{1}{6}\int_0^\infty \int_x^\infty e^{t_1 x} x^3\, e^{-y(1-t_2)}\,dy\,dx = \tfrac{1}{6}\int_0^\infty -x^3 e^{t_1 x}\left.\frac{e^{-y(1-t_2)}}{1-t_2}\right|_x^\infty dx$$

$$= \tfrac{1}{6}\int_0^\infty x^3 e^{t_1 x}\frac{e^{-x(1-t_2)}}{1-t_2}dx = \tfrac{1}{6(1-t_2)}\int_0^\infty x^3 e^{-x(1-t_1-t_2)}dx = \frac{1}{6(1-t_2)}\Gamma(4)\left(\frac{1}{1-t_1-t_2}\right)^4$$

$$= \frac{1}{6(1-t_2)}6\left(\frac{1}{1-t_1-t_2}\right)^4 = \frac{1}{(1-t_2)(1-t_1-t_2)^4}\ ; \quad for\ t_2 < 1 \quad and \quad t_1 + t_2 < 1$$

$$M(t_1) = M(t_1,0) = \frac{1}{(1-0)(1-t_1-0)^4} = \frac{1}{(1-t_1)^4};\quad for \quad t_1 < 1$$

$$M(t_2) = M(0,t_2) = \frac{1}{(1-t_2)(1-0-t_2)^4} = \frac{1}{(1-t_2)(1-t_2)^4} = \frac{1}{(1-t_2)^5};\quad for \quad t_2 < 1$$

[redacted]

Homework 2.1

1. Let $f(x,y)=12x^2y^3,\ \ 0<x<1,\ \ 0<y<1,$ zero elsewhere, be the pdf of X and Y.

a. Find $P\left(0<X<\frac{1}{2},\ 0<Y<\frac{1}{2}\right)$

b. Find $P(X<Y)$

c. Find $P(Y<X)$

d. Find $P(x+y\le 1)$

e. Find $E(XY)$

2. Let $f(x,y)=\frac{1}{6}e^{-\frac{x}{2}-\frac{y}{3}},\ \ 0<x<\infty,\ \ 0<y<\infty,$ zero elsewhere, be the pdf of X and Y. Let Z = X + Y.

a. Find the cdf of Z?
b. Find the pdf of Z?
c. Compute the $P(Z<3)$.
d. Compute the $P(Z>2)$

3. Given the joint pdf $f(x,y)$ *for* $x=1,\ 2$ *and* $y=1,\ 2,\ 3$ compute the following:

		Y		
		1	2	
X	1	$\frac{4}{15}$	$\frac{2}{15}$	
	2	$\frac{3}{15}$	$\frac{1}{15}$	
	3	$\frac{2}{15}$	$\frac{3}{15}$	

a. Find the marginal pdfs of X and Y.

b. Compute the $P(x=3,\ y=2)$ *and* $P(x\ge 2,\ y=1)$.

c. Compute the $E(XY)$

d. Compute the joint mgf, $M(t_1,t_2)$.

e. Compute the $E(XY)$ using the joint mgf.

f. Find the marginal mgfs for X and Y.

4. Let X and Y have the p.d.f. $f(x,y)=\frac{1}{4}$; $0<x<2$, $0<y<2$, zero elsewhere. Let the random variable Z be the sum of X and Y; i.e. Z=X+Y . Find the Cumulative Distribution Function (cdf) of Z .

5. Given the joint pdf of X and Y; $f(x,y)=\frac{1}{2}x^2e^{-y}$; $0<x<y<\infty$, compute the joint mgf of X and Y; $M(t_1,t_2)$, and the marginal mgfs of X and Y.

6. Given the joint pdf of X and Y, $f(x,y)=\frac{5}{16}xy^2$; $0<x<y<2$, find the $E(XY)$, and the marginal pdfs of X and Y .