Section 2.3 Conditional Distributions and Expectations

The conditional pdf of X given Y is given by $f\left(x|y\right)=\frac{f(x,y)}{f(y)}$; where $f\left(y\right)\neq 0$.

Discrete Variable: $\sum_x f\left(x|y\right)=\sum \frac{f(x,y)}{f(y)}=\frac{f(y)}{f(y)}=1$

Continuous Variable: $\int_x f\left(x|y\right)\,dx=\int_x \frac{f(x,y)}{f(y)}\,dx=\frac{f(y)}{f(y)}=1$

Since $f\left(x|y\right)$ is positive and sums or integrates to one, then $f\left(x|y\right)$ is a pdf.

Probabilities for discrete or continuous variables:

$$p\left(a<x<b\mid y\right)=\begin{cases}\sum_{x=a}^{b} f\left(x|y\right)\\ \int_a^b f\left(x|y\right)\,dx\end{cases}\text{; Similarly, } p\left(c<y<d\mid x\right)=\begin{cases}\sum_{y=c}^{d} f\left(y|x\right)\\ \int_c^d f\left(y|x\right)\,dy\end{cases}.$$

Conditional Expectation of $u\left(x\right)$ given y is $E\left(u\left(x\right)\mid y\right)=\begin{cases}\sum_{all\,x} u\left(x\right)f\left(x|y\right)\\ \int_{-\infty}^{\infty} u\left(x\right)f\left(x|y\right)\,dx\end{cases}.$

Conditional Mean of X given Y: $E\left(X\mid y\right)=\int_{-\infty}^{\infty} x\,f\left(x|y\right)\,dx$.

Conditional Variance of X given Y: $\sigma_{X|y}^2=E\left(\left[X-E\left(X|y\right)\right]^2\mid y\right)=E\left(X^2|y\right)-\left(E\left(X|y\right)\right)^2$

Example 1: $f\left(x,y\right)=2\,;\ 0<x<y<1$.

$f\left(x\right)=\int_x^1 2\,dy=2y\Big|_x^1=2-2x=2\left(1-x\right)$. Hence, $f\left(x\right)=2\left(1-x\right)\ ;\ 0<x<1$.

$f\left(y\right)=\int_0^y 2\,dx=2x\Big|_0^y=2y$. Hence, $f\left(y\right)=2y\ ;\ 0<y<1$.

$$f(x|y)=\frac{f(x,y)}{f(y)}=\frac{2}{2y}=\frac{1}{y}\ ;\ \text{Hence,}\quad f(x|y)=\frac{1}{y};\ 0<x<y\ \ and\ \ 0<y<1.$$

$$f(y|x)=\frac{f(x,y)}{f(x)}=\frac{2}{2(1-x)}=\frac{1}{(1-x)};\ \text{Hence,}\ f(y|x)=\frac{1}{(1-x)};\ x<y<1\ and\ 0<x<1.$$

Computing Probabilities:

$$P\left(0<x<\tfrac{1}{2}\,\middle|\,y=\tfrac{3}{4}\right)=\int_0^{\frac{1}{2}} f(x|y)\,dx=\int_0^{\frac{1}{2}}\frac{1}{y=\frac{3}{4}}\,dx=\frac{4x}{3}\Big|_0^{\frac{1}{2}}=\frac{4}{6}=\frac{2}{3}$$

$$P\left(0<x<\tfrac{1}{2}\right)=\int_0^{\frac{1}{2}} f(x)\,dx=\int_0^{\frac{1}{2}}2(1-x)\,dx=2x-x^2\Big|_0^{\frac{1}{2}}=1-\tfrac{1}{4}=\tfrac{3}{4}$$

Conditional Expectations:

$$E(X\,|\,y)=\int_0^y x\frac{1}{y}\,dx=\frac{x^2}{2}\frac{1}{y}\Big|_0^y=\frac{y}{2}\quad;\ \text{Hence,}\ E(X\,|\,y)=\frac{y}{2}\ ;\ 0<y<1.$$

$$E(Y|x)=\int_x^1 y\frac{1}{(1-x)}\,dy=\frac{y^2}{2}\frac{1}{(1-x)}\Big|_x^1=\frac{1-x^2}{2(1-x)}=\frac{x+1}{2};\ E(Y\,|\,x)=\frac{x+1}{2}\ ;\ 0<x<1$$

Conditional Variance: $\sigma^2_{X|y}=E(X^2|y)-\left(E(X|y)\right)^2$

$$E(X^2\,|\,y)=\int_0^y x^2\frac{1}{y}\,dx=\frac{x^3}{3}\frac{1}{y}\Big|_0^y=\frac{y^2}{3}$$

$$\sigma^2_{X|y}=E(X^2|y)-\left(E(X|y)\right)^2=\frac{y^2}{3}-\left(\frac{y}{2}\right)^2=\frac{y^2}{3}-\frac{y^2}{4}=\frac{y^2}{12};$$

Hence, $\sigma^2_{X|y}=\frac{y^2}{12};\ 0<y<1$

$$E(Y^2\,|\,x)=\int_x^1 y^2\frac{1}{(1-x)}\,dy=\frac{y^3}{3}\frac{1}{(1-x)}\Big|_x^1=\frac{1-x^3}{3(1-x)}=\frac{\cancel{(1-x)}(x^2+x+1)}{3\cancel{(1-x)}}=\frac{x^2+x+1}{3}$$

$$\sigma_{Y|x}^2 = E\left(Y^2|x\right) - \left(E\left(Y|x\right)\right)^2 = \frac{x^2+x+1}{3} - \left(\frac{x+1}{2}\right)^2 = \frac{x^2+x+1}{3} - \frac{x^2+2x+1}{4}$$

$$= \frac{4x^2+4x+4-3x^2-6x-3}{12} = \frac{x^2-2x+1}{12} = \frac{(x-1)^2}{12};$$

Hence, $\sigma_{Y|x}^2 = \frac{(x-1)^2}{12} \quad ; \quad 0 < x < 1$

Example 2: If X and Y are random variables of discrete type having pdf $f(x,y) = \frac{x+2y}{18}\,;\ (x,y) = (1,1),(1,2),(2,1),(2,2)$, determine the conditional mean and variance of Y given X.

Note: $f(x,y) = \frac{x+2y}{18}$ $for\ x = 1,2\ and\ y = 1,2$.

$$f(x) = \sum_{y=1}^{2} \frac{x+2y}{18} = \frac{(x+2)+(x+4)}{18} = \frac{2x+6}{18};\ x = 1,2 \quad \text{and} \quad f(x) = \frac{2x+6}{18};\ x = 1,2$$

$$f(y) = \sum_{x=1}^{2} \frac{x+2y}{18} = \frac{(1+2y)+(2+2y)}{18} = \frac{3+4y}{18};\ y = 1,2 \quad \text{and} \quad f(y) = \frac{3+4y}{18};\ y = 1,2$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\frac{x+2y}{18}}{\frac{2x+6}{18}} = \frac{x+2y}{2x+6}; \text{ and } f(y|x) = \frac{x+2y}{2x+6};\ y = 1,2\ \ and\ \ x = 1\ or\ 2$$

$$E(Y|x) = \sum_{y=1}^{2} y\frac{x+2y}{2x+6} = \sum_{y=1}^{2} \frac{xy+2y^2}{2x+6} = \frac{(x+2)+(2x+8)}{2x+6} = \frac{3x+10}{2x+6};\ where\ \ x = 1\ or\ 2$$

$$E(Y|x) = \frac{3x+10}{2x+6}\ ;\ where\ \ x = 1\ or\ 2$$

$$E(Y|x=1) = \frac{3+10}{2+6} = \frac{13}{8} \quad and \quad E(Y|x=2) = \frac{6+10}{4+6} = \frac{16}{10}$$

$$E\left(Y^2 \middle| x\right)=\sum_{y=1}^{2} y^2 \frac{x+2y}{2x+6}=\sum_{y=1}^{2} \frac{xy^2+2y^3}{2x+6}=\frac{(x+2)+(4x+16)}{2x+6}=\frac{5x+18}{2x+6};\ where\ \ x=1\ or\ \ 2$$

$$\sigma^2_{Y|x}=E\left(Y^2 \middle| x\right)-\left(E\left(Y \middle| x\right)\right)^2=\frac{5x+18}{2x+6}-\left(\frac{3x+10}{2x+6}\right)^2=\frac{x^2+6x+8}{(2x+6)^2};\ where\ \ x=1\ or\ \ 2$$

$$\sigma^2_{Y|x=1}=\frac{x^2+6x+8}{(2x+6)^2}=\frac{15}{64}\quad and\quad \sigma^2_{Y|x=2}=\frac{x^2+6x+8}{(2x+6)^2}=\frac{24}{100}$$

Now consider the joint pdf, $f(x,y)=\dfrac{x+2y}{18}\ \ for\ \ x=1,2\ \ and\ \ y=1,2$, in the form of a table.

find $f\left(y \middle| x=1\right)\ \ and\ \ f\left(y \middle| x=2\right)$ and compute their conditional mean and variance.

		Y		
		1	2	$f(x)$
X	1	$\frac{3}{18}$	$\frac{5}{18}$	$\frac{8}{18}$
	2	$\frac{4}{18}$	$\frac{6}{18}$	$\frac{10}{18}$
$f(y)$		$\frac{7}{18}$	$\frac{11}{18}$	

Y	$f\left(y \middle| x=1\right)$	$yP\left(y \middle| x=1\right)$	$y^2P\left(y \middle| x=1\right)$
1	$\frac{3/18}{8/18}=\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{5/18}{8/18}=\frac{5}{8}$	$\frac{10}{8}$	$\frac{20}{8}$
		$\mu_{Y\|x=1}=\frac{13}{8}$	$y^2P\left(y \middle| x=1\right)=\frac{23}{8}$
		$\sigma^2_{Y\|x=1}=\frac{23}{8}-\left(\frac{13}{8}\right)^2=\frac{184}{64}-\frac{169}{64}=\frac{15}{64}$	

Y	$f\left(y \middle| x=2\right)$	$yP\left(y \middle| x=2\right)$	$y^2P\left(y \middle| x=2\right)$
	$\frac{4/18}{10/18}=\frac{4}{10}$	$\frac{4}{10}$	$\frac{4}{10}$
2	$\frac{6/18}{10/18}=\frac{6}{10}$	$\frac{12}{10}$	$\frac{24}{10}$
		$\mu_{Y\|x=2}=\frac{16}{10}$	$y^2P\left(y \middle| x=2\right)=\frac{28}{10}$
		$\sigma^2_{Y\|x=2}=\frac{28}{10}-\left(\frac{16}{10}\right)^2=\frac{280}{100}-\frac{256}{100}=\frac{24}{10}$	

Example 3: Given the joint pdf of X and Y find $f(y|x=2)=\frac{f(x=2,y)}{f(x=2)}$ *and* $f(x|y=3)=\frac{f(x,y=3)}{f(y=3)}$ and compute their conditional mean and variance .

		Y			
		1	2	3	$f(x)$
X	1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
	2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{6}{10}$
$f(y)$		$\frac{2}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	

Y	$f(y\|x=2)$	$yP(y\|x=2)$	$y^2P(y\|x=2)$
1	$\frac{1}{10}\Big/\frac{6}{10}=\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{2}{10}\Big/\frac{6}{10}=\frac{2}{6}$	$\frac{4}{6}$	$\frac{8}{6}$
3	$\frac{3}{10}\Big/\frac{6}{10}=\frac{3}{6}$	$\frac{9}{6}$	$\frac{27}{6}$
		$\mu_{Y\|x=2}=\frac{14}{6}=\frac{7}{3}$	$y^2P(y\|x=2)=\frac{36}{6}=6$
		$\sigma^2_{Y\|x=2}=6-\left(\frac{7}{3}\right)^2=6-\frac{49}{9}=\frac{5}{9}$	

X	$f(x\|y=3)$	$xP(x\|y=3)$	$x^2P(x\|y=3)$
1	$\frac{2}{10}\Big/\frac{5}{10}=\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
2	$\frac{3}{10}\Big/\frac{5}{10}=\frac{3}{5}$	$\frac{6}{5}$	$\frac{12}{5}$
		$\mu_{X\|y=3}=\frac{8}{5}$	$y^2P(x\|y=3)=\frac{14}{5}$
		$\sigma^2_{X\|y=3}=\frac{14}{5}-\left(\frac{8}{5}\right)^2=\frac{70}{25}-\frac{64}{25}=\frac{6}{25}$	

Example 4: $f(x,y)=6y\ ;\ 0<y<x<1$.

$$f(y)=\int_y^1 6y\,dx=\ 6yx\Big|_y^1=6y(1-y)=6y-6y^2;\quad \text{Hence, } f(y)=6y-6y^2\ ;\ 0<y<1.$$

$$E(Y)=\int_0^1 y\left(6y-6y^2\right)dy=\int_0^1 6y^2-6y^3\,dy=2y^3-\tfrac{3}{2}y^4\Big|_0^1=1-\tfrac{3}{2}=\tfrac{1}{2};\ \text{Hence, } E(Y)=\tfrac{1}{2}$$

$$E(Y^2)=\int_0^1 y^2\left(6y-6y^2\right)dy=\int_0^1 6y^3-6y^4\,dy=\tfrac{3}{2}y^3-\tfrac{6}{5}y^5\Big|_0^1=\tfrac{3}{2}-\tfrac{6}{5}=\tfrac{15-12}{10}=\tfrac{3}{10};\ \text{Hence, } E(Y^2)=\tfrac{3}{10}$$

$$\sigma_Y^2=E(Y^2)-\left(E(Y)\right)^2=\frac{3}{10}-\left(\frac{1}{2}\right)^2=\frac{3}{10}-\frac{1}{4}=\frac{6-5}{20}=\frac{1}{20}\ ;\ \text{Hence, } \sigma_Y^2=\tfrac{1}{20}$$

On the average the random variable Y will be, $E(Y)=\frac{1}{2}$ with variance $\sigma_Y^2=\frac{1}{20}$. It means that Y as an estimator of μ is good but can we do better? Can we find another random variable, say V, so the mean or expected value of V is $E(V)=\frac{1}{2}$ but the variance of V is less than variance for Y, i.e $\sigma_V^2<\sigma_Y^2=\frac{1}{20}$?

In the search of such random variable consider the conditional pdf of Y given X ; i.e. $f\left(y|x\right)$.

$$f(x)=\int_0^x 6y\,dy=\frac{6}{2}y^2\Big|_0^x=3x^2;\quad \text{Hence, } f(x)=3x^2\ ;\ 0<x<1.$$

$$f\left(y|x\right)=\frac{f(x,y)}{f(x)}=\frac{6y}{3x^2}=\frac{2y}{x^2}\ ;\ \text{Hence, } f\left(y|x\right)=\frac{2y}{x^2};\ 0<y<x\ \ and\ \ 0<x<1.$$

$$E\left(Y\,|\,x\right)=\int_0^x y\frac{2y}{x^2}\,dy=\int_0^x\frac{2y^2}{x^2}\,dy=\frac{2y^3}{3x^2}\Bigg|_0^x=\frac{2x}{3};E\left(Y\,|\,x\right)=\frac{2x}{3}\ ;\ 0<x<1$$

Note: The $E\left(Y\,|\,x\right)=\dfrac{2x}{3}$ is a random variable say V; i.e $\mathrm{V}=\frac{2x}{3}$. The distribution of V is

$$G(\mathrm{V})=P(\mathrm{V}\le \mathrm{v})=P\left(\tfrac{2x}{3}\le \mathrm{v}\right)=P\left(x\le\tfrac{3\mathrm{v}}{2}\right)\ where\quad 0<V<\tfrac{2}{3}.$$

$$G(\mathrm{V})=\begin{cases}0 & ;\ \ \mathrm{v}\le 0\\ \int_0^{\frac{3\mathrm{v}}{2}}3x^2dx=x^3\Big|_0^{\frac{3\mathrm{v}}{2}}=\frac{27}{8}\mathrm{v}^3 & ;\ \ 0<\mathrm{v}<\frac{2}{3}\\ 1 & ;\ \ \mathrm{v}\ge\frac{2}{3}\end{cases}$$

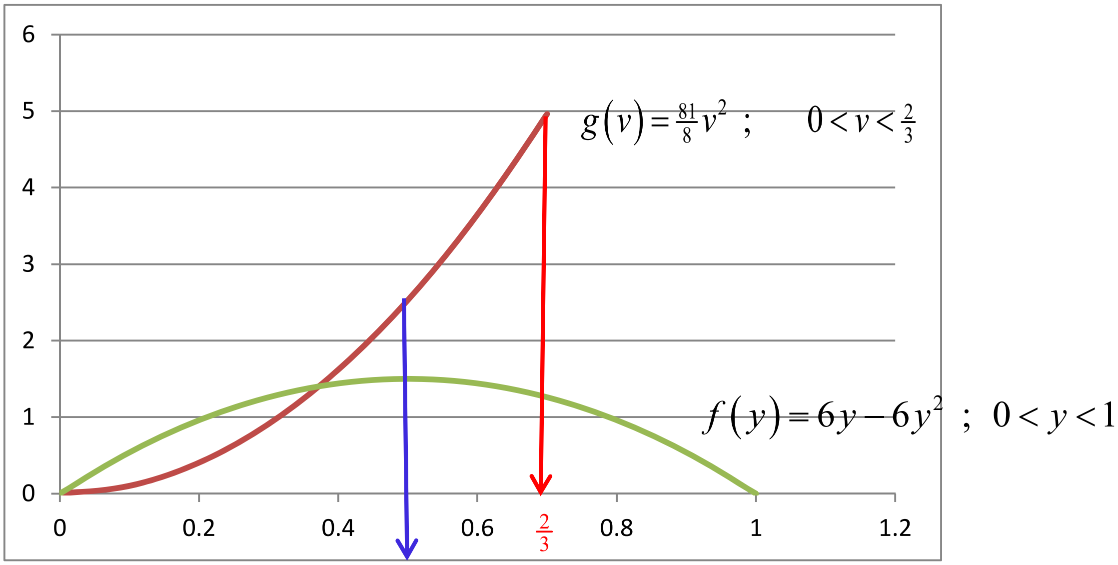
$$g(v)=\tfrac{81}{8}v^2\ ;\qquad 0<v<\tfrac{2}{3}.$$

$$E(V)=\int_0^{\frac{2}{3}} v\tfrac{81}{8}v^2\,dv=\int_0^{\frac{2}{3}}\tfrac{81}{8}v^3\,dv=\tfrac{81}{32}v^4\Big|_0^{\frac{2}{3}}=\tfrac{81}{32}\tfrac{16}{81}=\tfrac{1}{2};\ \text{Hence, } E(V)=\tfrac{1}{2}$$

$$E\left(V^2\right)=\int_0^{\frac{2}{3}} v^2\tfrac{81}{8}v^2\,dv=\int_0^{\frac{2}{3}}\tfrac{81}{8}v^4\,dv=\tfrac{81}{40}v^5\Big|_0^{\frac{2}{3}}=\tfrac{81}{40}\tfrac{32}{243}=\tfrac{4}{15};\ \text{Hence, } E\left(V^2\right)=\tfrac{4}{15}$$

$$\sigma_V^2=E\left(V^2\right)-\left(E(V)\right)^2=\frac{4}{15}-\left(\frac{1}{2}\right)^2=\frac{4}{15}-\frac{1}{4}=\frac{16-15}{60}=\frac{1}{60};\ \text{Hence, } \sigma_V^2=\tfrac{1}{60}$$

Note: $\sigma_{\mathrm{v}}^2=\frac{1}{60}<\ \sigma_y^2=\frac{1}{20}$



$E(Y)=\frac{1}{2}=E(V)$

$f(y)=6y-6y^2$; $0<y<1$	$g(v)=\frac{81}{8}v^2$; $0<v<\frac{2}{3}$
$E(Y)=\dfrac{1}{2}$ *and* $\sigma_Y^2=\dfrac{1}{20}$	$E(V)=E(Y\|x)=\dfrac{1}{2}$ *and* $\sigma_V^2=\dfrac{1}{60}$

[redacted]

Homework 2.3

1. If the pdf of X and Y is given by $f(x,y)$ *and* $0<x<y<1$, compute the following:

a. $f(y)=$; limits:

b. $E(Y)=$

c. $\sigma_Y^2=$

d. $f(x|y)=$ and ; limits:

e. $E(X|y)=$; limits:

f. $\sigma_{X|y}^2=$

2. If X and Y are random variables of discrete type having pdf

$$f(x,y)=\frac{3x+4y}{46}\ ;\ (x,y)=(0,2),(0,3),(1,2),(1,3)$$, determine the conditional mean and variance of Y given X. Note: $f(x,y)=\frac{3x+4y}{46}\ for\ x=0,1\ and\ y=2,3$.

3. Given the pdf $f(x,y)=6x\ ;\ 0<x<y<1$

a. Find $f(x),\ f(y),\ f(x|y)$, $and\ f(x|y)$.

b. Compute the $P\left(0<x<\frac{1}{4}\ \middle|\ y=\frac{2}{5}\right)$.

c. Compute the $E(X\mid y),\ E(X^2\mid y)$, and $\sigma^2_{X|y}$

4. Given the joint pdf of X and Y find $f(y|x=1)=\frac{f(x=1,y)}{f(x=1)}$ and $f(x|y=2)=\frac{f(x,y=2)}{f(y=2)}$ and compute their conditional mean and variance .

			Y		
		1	2	3	$f(x)$
X	1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
	2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{6}{10}$
$f(y)$		$\frac{2}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	

5. In Example 1 of the notes(section 2.3) $f(x)=2(1-x)\ ;\ 0<x<1$ and

$$f(y)=2y\ ;\ 0<y<1\ .$$

a. Find the $E(X),\ E(X^2)$, and σ^2_X .

b. In Example 1 of the notes(section 2.3) the $E(X\mid y)=\frac{y}{2}$. Let $V=\frac{y}{2}$

Find the pdf of V, the $E(V)$, and the σ^2_V.

c. Compare $E(V)\ and\ \sigma^2_V$ in part b to $E(X)\ and\ \sigma^2_X$ in part a and write your conclusion.