Section 2.4 The Correlation Coefficient

Let X and Y be two random variables. The correlation coefficient of X and Y is defined to be the covariance of X and Y, divided by the $\sqrt{\sigma_1^2\sigma_2^2}$. $\rho_{xy}=\dfrac{Cov(x,y)}{\sqrt{\sigma_1^2\sigma_2^2}}=\dfrac{E\big((x-\mu_1)(y-\mu_2)\big)}{\sigma_1\sigma_2}=\dfrac{E(XY)-\mu_1\mu_2}{\sigma_1\sigma_2}$

Note: $Cov(x,y)=E(XY)-\mu_1\mu_2$ *or* $E(XY)=Cov(x,y)+\mu_1\mu_2$

Example 1: Consider $f(x,y)=x+y$; *where* $0<x<1$ *and* $0<y<1$.

$f(x)=\int_0^1 x+y\,dy=x+\frac{y^2}{2}\Big|_0^1=x+\frac{1}{2};\quad 0<x<1$; $\quad f(x)=x+\frac{1}{2};\quad 0<x<1$

$E(X)=\int_0^1 x\left(x+\frac{1}{2}\right)dx=\frac{x^3}{3}+\frac{x^2}{4}\Big|_0^1=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$; $\quad E(X)=\frac{7}{12}=\mu_1$. Similarly, $E(Y)=\frac{7}{12}=\mu_2$

$E(X^2)=\int_0^1 x^2\left(x+\frac{1}{2}\right)dx=\frac{x^4}{4}+\frac{x^3}{6}\Big|_0^1=\frac{1}{4}+\frac{1}{6}=\frac{10}{24}=\frac{5}{12}$; $\quad E(X^2)=\frac{5}{12}$. Similarly, $E(Y^2)=\frac{5}{12}$

$\sigma_1^2=E(X^2)-(E(X))^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{60-49}{144}=\frac{11}{144}$; $\quad \sigma_1^2=\frac{11}{144}$. Similarly, $\sigma_2^2=\frac{11}{144}$

$$E(XY)=\int_0^1\int_0^1 xy(x+y)\,dxdy=\int_0^1 \frac{x^3y}{3}+\frac{x^2y^2}{2}\Big|_0^1 dy=\int_0^1 \frac{y}{3}+\frac{y^2}{2}dy=\frac{y^2}{6}+\frac{y^3}{6}\Big|_0^1=\frac{2}{6};\quad E(XY)=\frac{2}{6}$$

$$\rho_{xy}=\frac{Cov(x,y)}{\sqrt{\sigma_1^2\sigma_2^2}}=\frac{E(XY)-\mu_1\mu_2}{\sqrt{\sigma_1^2\sigma_2^2}}=\frac{\frac{2}{6}-\frac{7}{12}\frac{7}{12}}{\sqrt{\frac{11}{144}\frac{11}{144}}}=\frac{-\frac{1}{144}}{\frac{11}{144}}=-\frac{1}{11}=-0.090909;\quad \rho_{xy}=-0.090909$$

Note: $-1\le\rho\le 1$

If $\rho=1$, there is a line with equation $y=a+bx$, $b>0$. Hence, the probability for the distribution of X and Y comes from the points on the straight line, so $P(y=a+bx)=1$.
If $\rho=-1$, we have the same thing but $b<0$.

Question: Is there a line in the XY-plane such that the probability for X and Y tends to concentrate around this line?

If y has a linear conditional mean, that is, $E(Y|x)=\int_{-\infty}^{\infty} y\,f(y|x)\,dy=a+bx$, we can find the constants a, *and* b. Mathematics shows that $b=\rho\dfrac{\sigma_2}{\sigma_1}$ and $a=\mu_2-\rho\dfrac{\sigma_2}{\sigma_1}\mu_1$.

$$E(Y|x)=\int_{-\infty}^{\infty} y\,f(y|x)\,dy=a+bx=\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1+\rho\frac{\sigma_2}{\sigma_1}x=\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

Hence, the linear conditional mean of

$$E(Y|x)=\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$ and $$E(X|y)=\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2).$$

If Y has a linear conditional mean: $E(Y|x)=\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$, then the variance of the conditional distribution is $\sigma_{Y|x}^2=\sigma_2^2(1-\rho^2)$.

Results from the conditional variance of Y given X. Note: Since the variance is greater or equal to zero, it implies that

1. $(1-\rho^2)\ge 0 \quad\Rightarrow\quad \rho^2\le 1 \quad\Rightarrow\quad -1\le\rho\le 1$
2. $\sigma_{Y|x}^2=\sigma_2^2(1-\rho^2);\quad Note: If\ \ \rho=0,\ \ then\ \ \sigma_{Y|x}^2=\sigma_2^2$
3. $If\ \rho>0,\ then\ \ \sigma_{Y|x}^2<\sigma_2^2$
4. Note: $\rho\frac{\sigma_2}{\sigma_1}\times\rho\frac{\sigma_1}{\sigma_2}=\rho^2$
5. If ρ^2 is near one then $\sigma_{Y|x}^2$ is relatively small, meaning there is a high concentration of the probability for this conditional distribution near the conditional mean of Y given X; $E(Y|x)=\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$.

   Note: If the random variables are discrete the results will still hold.

Example 2: Given the conditional means $E(Y|x)=4x+3\ \ and\ \ E(X|y)=\frac{1}{16}y-3$, find $\mu_1,\ \mu_2\ ,\ and\ \rho$.

$E(E(Y\|x))=E(4x+3)$	$E(E(X\|y))=E(\frac{1}{16}y-3)$
$\mu_2=4\mu_1+3$	$\mu_1=\frac{1}{16}\mu_2-3$

Hence, $\mu_2=4(\frac{1}{16}\mu_2-3)+3=\frac{1}{4}\mu_2-12+3\Rightarrow\mu_2=\frac{1}{4}\mu_2-9\Rightarrow 4\mu_2=\mu_2-36\Rightarrow\mu_2=-12$

$\mu_1=\frac{1}{16}(-12)-3\Rightarrow\mu_1=-\frac{15}{4}$

Note: $E(Y|x)=a_2+b_2x\ \ and\ \ E(Y|y)=a_1+b_1y \quad\Rightarrow\quad \rho^2=b_1b_2$. Hence, $\rho^2=4\times\frac{1}{16}=\frac{1}{4}$

$\rho=\pm\sqrt{\frac{1}{4}}\ \Rightarrow\ \rho=\pm\frac{1}{2}$. Since the coefficients of X and Y are both positive, $\rho=\frac{1}{2}$.

Example 3: Consider the pdf $f(x,y)=e^{-y};\ \ 0<x<y<\infty$.

$$M(t_1,t_2)=E\left(e^{t_1x+t_2y}\right)=\int_0^\infty\int_x^\infty e^{t_1x+t_2y}e^{-y}\,dy\,dx=\int_0^\infty\int_x^\infty e^{t_1x}e^{-y(1-t_2)}\,dy\,dx=\int_0^\infty e^{t_1x}\left(-\tfrac{1}{1-t_2}e^{-y(1-t_2)}\Big|_x^\infty\right)dx$$

$$=\int_0^\infty e^{t_1x}\,\tfrac{e^{-x(1-t_2)}}{(1-t_2)}\,dx=\int_0^\infty \tfrac{e^{-x(1-t_1-t_2)}}{(1-t_2)}\,dx=\tfrac{1}{(1-t_2)}\Gamma(1)\left(\tfrac{1}{(1-t_1-t_2)}\right)^1=\tfrac{1}{(1-t_2)(1-t_1-t_2)}$$

Hence, $M(t_1,t_2)=\frac{1}{(1-t_2)(1-t_1-t_2)}\ ;\ where\ \ t_2<1\ \ and\ \ t_1+t_2<1$. Now, using this mgf verify that $\mu_1=1,\ \mu_2=2,\ \sigma_1^2=1,\ \sigma_2^2=2,\ \ and\ \ E\left((X-\mu_1)(Y-\mu_2)\right)=1$.

$M(t_1)=M(t_1,0)=\frac{1}{(1-0)(1-t_1-0)}=\frac{1}{1-t_1}=(1-t_1)^{-1}$. $\qquad \mu_1=M'(t_1=0)=-1(1-t_1)^{-2}(-1)\Big|_{t_{1=0}}=1$

$M(t_2)=M(0,\,t_2)=\frac{1}{(1-t_2)(1-0-t_2)}=\frac{1}{(1-t_2)^2}=(1-t_2)^{-2}$. $\qquad \mu_2=M'(t_2=0)=-2(1-t_2)^{-3}(-1)\Big|_{t_{2=0}}=2$

$E(X^2)=M''(t_1)=-2(1-t_1)^{-3}(-1)\Big|_{t_{1=0}}=2$ and $\qquad \sigma_1^2=E(X^2)-(\mu_1)^2=2-(1)^2=1$

$E(Y^2)=M''(t_2)=2(-3)(1-t_1)^{-4}(-1)\Big|_{t_{2=0}}=6$ and $\qquad \sigma_2^2=E(Y^2)-(\mu_2)^2=6-(2)^2=2$

To compute the expected value of , $E(XY)=\dfrac{\partial^2M(0,0)}{\partial t_1^1\partial t_2^1}$, first we take the derivative w.r.t. t_1; i.e.

$\dfrac{\partial}{\delta t_1}\left((1-t_2)^{-1}(1-t_1-t_2)^{-1}\right)=(1-t_2)^{-1}(-1)(1-t_1-t_2)^{-2}(-1)=(1-t_2)^{-1}(1-t_1-t_2)^{-2}$. Next, we use the derivative w.r.t. t_1 to take the derivative with respect to t_2; i.e

$$\frac{\partial}{\delta t_2}\left((1-t_2)^{-1}(1-t_1-t_2)^{-2}\right)=(-1)(1-t_2)^{-2}(-1)(1-t_1-t_2)^{2}+(1-t_2)^{-1}(-2)(1-t_1-t_2)^{-3}(-1)$$

Hence, $E(XY)=\dfrac{\partial^2M(0,0)}{\partial t_1^1\partial t_2^1}=(1-t_2)^{-2}(1-t_1-t_2)^{2}+(1-t_2)^{-1}(2)(1-t_1-t_2)^{-3}\Big|_{t_1=t_2=0}=1+2=3$

Hence, $Cov(X,Y)=E\left((X-\mu_1)(Y-\mu_2)\right)=E(XY)-\mu_1\mu_2=3-(1\times 2)=1$

Finally, $\rho_{xy}=\dfrac{Cov(x,y)}{\sqrt{\sigma_1^2\sigma_2^2}}=\dfrac{1}{\sqrt{1\times 2}}=\dfrac{1}{\sqrt{2}}$

Example 4: Consider the joint pdf $f(x,y)=e^{-y};\ 0<x<y<\infty$. From **Example 3** we know that

$\mu_1=1,\ \mu_2=2,\ \sigma_1^2=1,\ \sigma_2^2=2,\ and\ \ \rho=\frac{1}{\sqrt{2}}$; $f(x)=e^{-x};\ 0<x<\infty\ and\ f(y)=ye^{-y};\ 0<y<\infty$

The conditional pdfs are $f(x|y)=\frac{1}{y};\ 0<x<y\quad and\quad f(y|x)=\frac{e^{-y}}{e^{-x}};\ x<y<\infty$.

$E(X|y)=\int_0^y \frac{x}{y}dx=\frac{x^2}{2y}\Big|_0^y=\frac{y}{2}$; Hence, $E(X|y)=\frac{y}{2}\ ;\quad 0<y<\infty$

$$E(Y|x)=\int_x^\infty \frac{ye^{-y}}{e^{-x}}dy=\frac{1}{e^{-x}}\left[-ye^{-y}\Big|_x^\infty-\int_x^\infty -e^{-y}dy\right]=\frac{1}{e^{-x}}\left[0+xe^{-x}-e^{-x}\Big|_x^\infty\right]=\frac{1}{e^{-x}}\left[0+xe^{-x}+e^{-x}\right]=x+1$$

$E(Y|x)=x+1\ ;\ 0<x<\infty$

Since the conditional means are linear we can use $\rho^2=b_1b_2$ to find the correlation ρ.

$\rho^2=1\times\frac{1}{2}=\frac{1}{2}\Rightarrow\rho=\pm\sqrt{\frac{1}{2}}\Rightarrow\rho=\frac{1}{\sqrt{2}}$. Same answer as in **Example 3**.

Now use the information given in this example to compute

$E(Y|x)=\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)=2+\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{1}(x-1)=2+x-1=x+1;$ $\quad E(Y|x)=x+1$

$E(X|y)=\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)=1+\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(y-2)=1+\frac{1}{2}(y-2)=1+\frac{y}{2}-1=\frac{y}{2};$ $\quad E(X|y)=\frac{y}{2}$

[redacted]

Homework 2.4

1. If the pdf of X and Y is given by $f(x,y)$ and $0<x<y<1$, give the formulas for the following.
   a. $\rho_{xy} =$
   b. $E\left(E\left(X|y\right)\right) =$
   c. $Cov(X,Y) =$
   d. If X and Y have linear conditional means, then $E\left(X|y\right) =$
   $E\left(Y|x\right) =$
   e. If X has a linear conditional mean, then $\sigma^2_{X|y} =$

2. Given the conditional means $E\left(Y|x\right) = -2x-1$ and $E\left(X|y\right) = -\frac{1}{5}y+2$, find μ_1, μ_2 , and ρ.

3. Consider the pdf $f(x,y) = \frac{1}{2}x^2e^{-y};\ 0<x<y<\infty$. Compute the correlation between X and Y.

4. Consider the joint pdf $f(x,y) = 4e^{-2y};\ 0<x<y<\infty$.
   Given $\mu_1 = \frac{1}{2},\ \mu_2 = 1,\ \sigma_1^2 = \frac{1}{4},\ \sigma_2^2 = \frac{1}{2},\ E(XY) = \frac{3}{4}$, $E\left(X|y\right) = \frac{y}{2},\ and\ E\left(Y|x\right) = x+\frac{1}{2}$.
   a. Compute $\rho_{xy} =$
   b. Use the conditional linear means, $E\left(X|y\right) = \frac{y}{2}$ and $E\left(Y|x\right) = x+\frac{1}{2}$ to compute ρ_{xy} .
   c. Use the information given in the problem to compute the
   $$E\left(Y|x\right) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}\left(x-\mu_1\right) =$$
   $$E\left(X|y\right) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(y-\mu_2\right) =$$