Chapter 3 Some Special Distributions

Section 3.3 The Gamma and Chi-Square Distributions

Gamma: From calculus we know that $\int_0^\infty y^{\alpha-1} e^{-y}\, dy$ exists for $\alpha > 0$ and that the value of the integral is a positive number. $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$ is called the Gamma function.

If $\alpha = 1$ then $\Gamma(\alpha = 1) = \int_0^\infty y^{1-1} e^{-y}\, dy = \int_0^\infty e^{-y}\, dy = 1$.

Integration By Parts

If $\alpha > 1$ then $\Gamma(\alpha) = (\alpha - 1)\int_0^\infty y^{\alpha-2} e^{-y}\, dy = (\alpha-1)\int_0^\infty y^{(\alpha-1)-1} e^{-y}\, dy = (\alpha-1)\Gamma(\alpha-1)$.

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Accordingly, if α is a positive integer greater than 1, then
$\Gamma(\alpha) = (\alpha-1)(\alpha-2)(\alpha-3)\ldots(3)(2)(1)\Gamma(1) = (\alpha-1)!$

Note: Since $\Gamma(1) = 1$, suggests that $0! = 1$.

Derivation of the Gamma Distribution

Now consider the Gamma function, $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$, and the change of variable $Y = \frac{X}{\beta}$; where $\beta > 0$.

$$\Gamma(\alpha) = \int_0^\infty \left(\tfrac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} \tfrac{d}{dx}\left(\tfrac{x}{\beta}\right) dx = \int_0^\infty \left(\tfrac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} \tfrac{1}{\beta}\, dx = \tfrac{1}{\beta^\alpha}\int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx$$

Now divide both sides by $\Gamma(\alpha)$. We have $1 = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx$. Since, $\alpha > 0,\ \beta > 0$, and $\Gamma(\alpha) > 0$, we have a new probability density function (pdf) called the Gamma distribution, $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$; $0 < x < \infty$. So, the random variable X has a Gamma distribution with parameters α and β and it's denoted by $X \sim Gamma(\alpha, \beta)$.

Find the Moment Generating Function, mgf, of the Gamma Distribution.

$$M(t)=E\left(e^{tx}\right)=\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}e^{tx}x^{\alpha-1}e^{-\frac{x}{\beta}}\,dx=\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x\left(\frac{1}{\beta}-t\right)}\,dx=\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x\left(\frac{1-\beta t}{\beta}\right)}\,dx$$

$$M(t)=\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)\int_0^\infty x^{\alpha-1}e^{-x\left(\frac{1-\beta t}{\beta}\right)}\,dx=\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)\Gamma(\alpha)\left(\frac{\beta}{1-\beta t}\right)^\alpha=\left(\frac{1}{\beta^\alpha}\right)\left(\frac{\beta^\alpha}{(1-\beta t)^\alpha}\right)=\frac{1}{(1-\beta t)^\alpha}$$

$M(t)=\frac{1}{(1-\beta t)^\alpha}$ or $(1-\beta t)^{-\alpha}$ where $(1-\beta t)>0 \Rightarrow t<\frac{1}{\beta}$.

Mean and Variance of the Gamma Distribution

$M'(t)=-\alpha\left(1-\beta t\right)^{-\alpha-1}(-\beta)$ and $M'(0)=-\alpha\left(1-\beta .0\right)^{-\alpha}(-\beta)=\alpha\beta \quad \Rightarrow \mu=\alpha\beta$.

$M''(t)=\alpha\beta\left[-(\alpha+1)\left(1-\beta t\right)^{-\alpha-2}(-\beta)\right]$ and

$M''(0)=\alpha\beta\left[-(\alpha+1)\left(1-\beta .0\right)^{-\alpha-2}(-\beta)\right]=\alpha(a+1)\beta^2$.

$\sigma^2=M''(0)-\left(M'(0)\right)^2=\alpha(\alpha+1)\beta^2-(a\beta)^2=\alpha^2\beta^2+\alpha\beta^2-\alpha^2\beta^2=\alpha\beta^2 \Rightarrow \sigma^2=\alpha\beta^2$.

Chi-Square Distribution

Now, let $\alpha=\frac{r}{2}$ where r is a positive integer and $\beta=2$. $f(x)=\frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\frac{r}{2}}}x^{\frac{r}{2}-1}e^{-\frac{x}{2}}$; $0<x<\infty$.

$M(t)=\frac{1}{(1-2t)^{\frac{r}{2}}}$ or $(1-2t)^{-\frac{r}{2}}$ where $t<\frac{1}{2}$.

X has a Chi-square distribution with r degrees of freedom and it's denoted by $X\sim\chi^2_{(r)}$.

$\mu=\left(\frac{r}{2}\right)2=r$ and $\sigma^2=\left(\frac{r}{2}\right)2^2=2r$

Example: Let $\chi^2_{(10)}$ then by Table II of Appendix C(page 658), with $r=10$,

$$P(3.25\le X\le 20.5)=P(X\le 20.5)-P(X\le 3.25)=0.975-0.025=0.95$$

Table II
Chi-Square Distribution

The following table presents selected quantiles of chi-square distribution, i.e., t
values x such that

$$P(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2}\, dw,$$

for selected degrees of freedom r.

	$P(X \leq x)$							
r	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217
13	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289
23	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642
27	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892

Example: Consider the Gamma Distribution where $X \sim Gamma\left(\frac{r}{2},\beta\right)$;

$f(x)=\frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}}x^{\frac{r}{2}-1}e^{-\frac{x}{\beta}}$; $0<x<\infty$. If $Y=\frac{2X}{\beta}$, find the pdf of Y.

Start from $G(Y)=\mathrm{P}(Y\le y)=\mathrm{P}\left(\frac{2X}{\beta}\le y\right)=\mathrm{P}\left(X\le\frac{\beta y}{2}\right)=\int_0^{\frac{\beta y}{2}}\frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}}x^{\frac{r}{2}-1}e^{-\frac{x}{\beta}}\,dx$

$$g(y)=G'(Y)=\frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}}\left(\frac{\beta y}{2}\right)^{\frac{r}{2}-1}e^{-\frac{\beta y}{2}\cdot\frac{1}{\beta}}\cdot\frac{d}{dy}\left(\frac{\beta y}{2}\right)-0=\frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}}\left(\frac{\beta y}{2}\right)^{\frac{r}{2}-1}e^{-\frac{\beta y}{2}\cdot\frac{1}{\beta}}\cdot\left(\frac{\beta}{2}\right)$$

$$=\frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}}\left(\frac{\beta}{2}\right)^{\frac{r}{2}-1}y^{\frac{r}{2}-1}e^{-\frac{y}{2}}\cdot\left(\frac{\beta}{2}\right)=\frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}}\left(\frac{\beta}{2}\right)^{\frac{r}{2}}y^{\frac{r}{2}-1}e^{-\frac{y}{2}}=\frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}}\frac{\beta^{\frac{r}{2}}}{2^{\frac{r}{2}}}y^{\frac{r}{2}-1}e^{-\frac{y}{2}}$$

$$=\frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\frac{r}{2}}}y^{\frac{r}{2}-1}e^{-\frac{y}{2}}$$

$g(y)=\frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\frac{r}{2}}}y^{\frac{r}{2}-1}e^{-\frac{y}{2}}$; $0<y<\infty$. Hence, $Y\sim\chi^2_{(r)}$

Example: Consider the function $f(x)=\frac{1}{\Gamma(4)\,3^4}x^3e^{-\frac{x}{3}}$; $0<x<\infty$ **.**

Compute the $P(4.1<X<20.043)$ **.**

Note: if $X\sim Gamma(\alpha=4,\beta=3)$ and let $Y=\frac{2X}{\beta}$ then $Y\sim\chi^2_{(8)}$**.**

Hence,

$$P(4.1<X<20.043)=P\left(4.1\left(\tfrac{2}{3}\right)<\tfrac{2}{\beta}X<20.043\left(\tfrac{2}{3}\right)\right)=P(2.733<Y<13.362)$$

$$=P(Y<13.362)-P(Y<2.733)=0.9-0.05=0.85\,\mathbf{.}$$

Homework 3.3

1. For each distribution provide the pdf, mean, variance, and mgf. Make sure you include the limits for each pdf. a. Gamma b. Chi-Square

2. Derive the Gamma Distribution.

3. Given X has a Gamma Distribution, find $M(t)$ and then compute $E(X)$ *and* σ^2.

4. Consider the Gamma Distribution where $X \sim Gamma\left(\frac{r}{2}, \beta\right)$;

   $f(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{\beta}}$; $0 < x < \infty$. If $Y = \frac{2X}{\beta}$, find the pdf of Y.

5. Consider the function $f(x) = \frac{1}{\Gamma(3)\,4^3} x^2 e^{-\frac{x}{4}}$; $0 < x < \infty$.

   Compute the $P(2.474 < X < 21.29)$.

6. Consider the pdf $f(x) = 2x$; $0 < x < 1$. Find the cdf of $Y = -6\ln X$. What is the pdf of Y ? Did you recognize the pdf of Y?