Topological Data Analysis and Persistence Theory

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Introduction

The goal of this conference is to introduce many aspects of Topological Data Analysis (TDA) and Persistence Theory. We will start with the basic constructions and fundamental results and work our way up to recent research progress and open problems. There will be ten lectures, three group work and discussion sessions, three lab sessions, and three open discussion sessions.

First a disclaimer. The subject at hand is already much larger than what can be covered in the time allotted. For example, I will focus on persistent homology and not discuss other important tools in TDA such as the Mapper algorithm and Reeb graphs. Many interesting theoretical results and exciting applications will be omitted. Hopefully, this introduction will encourage participants to discover other parts of this subject and make it easier for them to do so.

The main part of the conference is a sequence of ten lectures, which I am looking forward to be giving. Each of these will tackle one topic and together they are roughly equally divided between TDA and persistence theory. The lectures will be supplemented by three short sets of exercises which may be done with pencil and paper. Time is allotted for participants to work on these in small groups. An important aim of these exercises in addition to learning the material is to get to know some of the other participants and to have fun! Perhaps the second main part of the conference is a sequence of three lab sessions in which participants will learn how to perform TDA computations using the programming language R. These sessions will be led by Iryna Hartsock and John Bush.

Historical background

The roots of persistent homology may already be found in the work of Marston Morse [Mor25]. As observed by Raoul Bott [Bot88], the following is a central result of Morse's theory. Let f be a Morse function on a manifold M. Morse proved that not only is the ith Betti number a lower bound on the number of critical points of f of degree i, but the excess critical points may be partitioned into pairs that differ in degree by 1. What Morse did not show is that this pairing is canonical and that if we consider the corresponding pairs of critical values we obtain what is now called the persistence diagram of f, for which there is an efficient algorithm and which is useful in applications.

Vanessa Robins [Rob99] defined persistent homology groups and studied their ranks, which she called persistent Betti numbers. Herbert Edelsbrunner, David Letscher, and Afra Zomorodian gave an efficient algorithm for computing the persistence diagram and showed that it could be used to obtain the persistent Betti numbers. Zomorodian and Gunnar Carlsson [ZC05] used a structure theorem to justify the pairing of critical values in the persistence diagram. An early application of persistent homology in topological data analysis was the study of the space of natural images by Carlsson, Tigran Ishkhanov, Vin de Silva, and Zomorodian [CIdSZ08].

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